Pergamon

Int. J. Solids Structures Vol. 33, No. 14, pp. 1961–1994, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00143–3

MASONRY-LIKE SOLIDS WITH BOUNDED COMPRESSIVE STRENGTH

MASSIMILIANO LUCCHESI
Università di Chieti, Viale Pindaro 42, 65100 Pescara, Italy

and

CRISTINA PADOVANI and NICOLA ZANI
Istituto CNUCE-CNR, Via Santa Maria 36, 56100 Pisa, Italy

(*Received* 16 *March* 1995; *in revised form* 29 *May* 1995)

Abstract—This paper proposes a constitutive equation for masonry-like materials with bounded compressive strength. The general properties of this equation are proved and its solution is explicitly calculated. Subsequently, a numerical method is proposed in order to solve the equilibrium problem of masonry-like solids with bounded compressive strength. In particular the derivative of the stress with respect to the total strain is calculated; this derivative will be used for calculating the tangent stiffness matrix and then for solving the non-linear system, obtained with the discretisation into finite elements via the Newton–Raphson method. Finally, this numerical method is applied to the study of Mosca's bridge in Turin and to the study of a three-dimensional circular reduced arch subjected to its own weight and a vertical load distributed along the extrados.

1. INTRODUCTION

Studying equilibrium problems of masonry solids requires above all a suitable choice for the constitutive equation of the material. Since these materials are in general very heterogeneous and their behaviour depends heavily on construction techniques, it appears extremely difficult to formulate constitutive equations general enough for application to all kinds of construction, but yet simple enough to permit solution of the main boundary-value problems encountered in practice.

In several cases of masonry arches and vaults it seems quite realistic to suppose the material to be elastic, isotropic, non-resistant to traction and infinitely resistant to compression. More precisely, one supposes that the infinitesimal strain tensor $\mathbf{E}$ is the sum of an elastic part $\mathbf{E}^e$ and an inelastic part $\mathbf{E}^a$ and that the stress $\mathbf{T}$, negative semi-definite, depends linearly and isotropically on $\mathbf{E}^e$. Moreover, it is required that $\mathbf{E}^a$, interpreted as fracture strain, be positive semi-definite and orthogonal to $\mathbf{T}$. Thus one obtains a hyper-elastic material, usually called masonry-like, which has been studied by many authors. In particular Del Piero (1989) detailed both the properties of the constitutive equations and some conditions concerning the admissibility of loads. Lucchesi *et al.* (1994b) have generalised the constitutive equation proposed by Del Piero (1989), in order to take into account that masonries are weakly resistant to traction, by allowing principal stresses to reach a value $\sigma^t > 0$, in correspondence to which, fracture strains arise. σ^t, called fracture stress, is a material constant which must be experimentally determined. The application of this constitutive equation to masonry solids is rather questionable because the two parts of the body separated by the fracture continue to transmit tensile stresses. Nevertheless, in some cases the use of this constitutive relation can furnish information useful for the design, as proved by Rossi and Sassu (1994), who have studied a masonry panel subjected to seismic load.

Lucchesi (1994a, 1994b) have proposed a numerical procedure for solving equilibrium problems of masonry structures via the finite element method. The application of this procedure to the study of arches (Lucchesi *et al.*, 1993), and vaults (Lucchesi *et al.*, 1994c) subjected to concentrated loads shows that by using the constitutive equation of masonry-like materials, the mechanism of collapse and the corresponding multiplier can be determined very accurately.

The hypothesis that masonries are infinitely resistant to compression, although not conservative, is justified in many situations by the fact that for particular load conditions collapse occurs as a consequence of the mechanisms activated when the compressive stress in the whole structure is inferior to its limit value. On the contrary, if an arch is subjected to its own weight and a vertical load distributed along the extrados, the line of thrust remains within the internal part of the arch and hinges do not form. Therefore, if the material were infinitely resistant to compression, the load could be increased indefinitely without ever reaching collapse. This result has suggested generalising the constitutive equation of masonry-like materials by introducing a bound to the compressive strength. More precisely, we suppose that, besides a positive semi-definite inelastic strain, a negative semi-definite inelastic strain may occur and be interpreted as crushing strain. Moreover, we suppose that the Cauchy stress has eigenvalues within the range $[-\sigma^c, \sigma^t]$, where σ^c, like σ^t, is a material constant called crushing stress, which must be experimentally determined. We obtain a hyperelastic non-linear material which will be called BCS masonry-like material (masonry-like material with bounded compressive strength) or, more simply, BCS material.

In Section 2 we investigate some relevant properties of the constitutive equation of BCS materials and explicitly calculate its solution for the two- and three-dimensional cases.

Then, in Section 3 we consider the boundary-value problem and prove that under suitable conditions the solution of the global and incremental equilibrium problems is unique in terms of stress: the former generalises the results obtained by Giaquinta and Giusti (1985) and Anzellotti (1985) for masonry-like materials, the latter is necessary in order to justify the use of incremental numerical techniques. In fact, it is proven that two different load processes applied to a structure made up of a BCS material, having the same final value and corresponding to which the solution of the boundary value problem exists, produce the same stress field. This result is a direct consequence of the elasticity of BCS materials, which shows the different behaviour of these materials and elastic–plastic ones.

The explicit solutions to the equilibrium problem of a circular ring and of a spherical container subjected to two uniform radial pressures p_i and p_e, acting, respectively, on the inner and outer boundary, are calculated in Section 4.

In Section 5 we calculate the derivative of the stress with respect to the total strain. This is needed in order to calculate the tangent stiffness matrix used in solving equilibrium problems with finite elements via the Newton–Raphson method. The numerical method we obtain generalises the method presented by Lucchesi *et al.* (1994a, 1994b) for materials infinitely resistant to compression.

In Section 6 we compare the exact solution for the circular ring calculated in Section 4 with the numerical solution. Moreover, we consider Mosca's bridge, in Turin, and calculate the line of thrust by comparing the results of our analysis with those of Castigliano (1879). Finally, we apply the numerical method to the study of a three-dimensional circular reduced arch subjected to its own weight and a vertical load distributed along the extrados. The distributed load is progressively increased until collapse is reached; then, the determination of the line of thrust and the position of hinges at the instant of collapse allows easy interpretation of the results of the numerical analysis.

2. THE CONSTITUTIVE EQUATION

In this section we describe the main properties of the constitutive equation of masonry-like materials with bounded compressive strength. Let us first present some notation: let $\mathscr{V}$ be a three-dimensional linear space and Lin the space of all linear applications of $\mathscr{V}$ into $\mathscr{V}$, equipped with the inner product $\mathbf{A} \cdot \mathbf{B} = \text{tr}(\mathbf{A}^T\mathbf{B})$, $\mathbf{A}$, $\mathbf{B} \in \text{Lin}$, with $\mathbf{A}^T$ the transpose of $\mathbf{A}$. Let us indicate as Sym, Sym^+ and Sym^- the subsets of Lin constituted by symmetric, symmetric positive semi-definite and symmetric negative semi-definite tensors, respectively.

Let us assume that the infinitesimal strain $\mathbf{E}$ is the sum of an elastic part $\mathbf{E}^e$ and of two mutually orthogonal inelastic parts $\mathbf{E}^t$ and $\mathbf{E}^c$ called fracture strain and crushing strain, respectively. $\mathbf{E}^t$ is positive semi-definite and $\mathbf{E}^c$ is negative semi-definite:

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^t + \mathbf{E}^c, \tag{1}$$

$$\mathbf{E}^t \in \mathrm{Sym}^+, \tag{2}$$

$$\mathbf{E}^c \in \mathrm{Sym}^-, \tag{3}$$

$$\mathbf{E}^t \cdot \mathbf{E}^c = 0. \tag{4}$$

Moreover, we suppose that the Cauchy stress $\mathbf{T}$ depends linearly on $\mathbf{E}^e$,

$$\mathbf{T} = \mathbb{C}[\mathbf{E}^e], \tag{5}$$

where $\mathbb{C}$ is a definite positive fourth-order tensor. We shall limit ourselves to the particular case in which the dependence of $\mathbf{T}$ on $\mathbf{E}^e$ is isotropic, and thus, $\mathbb{C}[\mathbf{E}^e] = 2\mu\mathbf{E}^e + \lambda \operatorname{tr}(\mathbf{E}^e)\mathbf{I}$, where the Lamé moduli μ and λ of the material satisfy the inequalities:

$$\mu > 0, \quad 2\mu + 3\lambda > 0. \tag{6}$$

Finally, we assume the existence of two positive material constants σ^t and σ^c, namely the maximum resistance, respectively, to traction and compression, such that:

$$\mathbf{T} - \sigma^t\mathbf{I} \in \mathrm{Sym}^-, \tag{7}$$

$$\mathbf{T} + \sigma^c\mathbf{I} \in \mathrm{Sym}^+, \tag{8}$$

$$(\mathbf{T} - \sigma^t\mathbf{I}) \cdot \mathbf{E}^t = (\mathbf{T} + \sigma^c\mathbf{I}) \cdot \mathbf{E}^c = 0. \tag{9}$$

When the crushing stress σ^c goes to infinity, relations (1)–(9) reduce to the constitutive equation of the masonry-like material infinitely resistant to compression studied by Lucchesi *et al.* (1994b).

Given $\mathbf{E} \in \mathrm{Sym}$, the Lamé moduli and the constants σ^c and σ^t, the constitutive equations (1)–(5) and (7)–(9) have a unique solution $(\mathbf{T}, \mathbf{E}^t, \mathbf{E}^c)$. In fact, we shall explicitly construct this solution; in order to prove its uniqueness, let $(\mathbf{T}_1, \mathbf{E}^t_1, \mathbf{E}^c_1)$ and $(\mathbf{T}_2, \mathbf{E}^t_2, \mathbf{E}^c_2)$ be two different solutions. For the elastic parts we have $\mathbf{E}^e_1 = \mathbf{E} - \mathbf{E}^c_1 - \mathbf{E}^t_1$ and $\mathbf{E}^e_2 = \mathbf{E} - \mathbf{E}^c_2 - \mathbf{E}^t_2$ and thus

$$\mathbb{C}^{-1}[\mathbf{T}_1 - \mathbf{T}_2] = \mathbf{E}^e_1 - \mathbf{E}^e_2 = \mathbf{E}^c_2 - \mathbf{E}^c_1 + \mathbf{E}^t_2 - \mathbf{E}^t_1.$$

By virtue of eqns (7)–(9) and the positive definiteness of $\mathbb{C}$, we can write

$$\begin{aligned} 0 \leqslant (\mathbf{T}_1 - \mathbf{T}_2) \cdot \mathbb{C}^{-1}[\mathbf{T}_1 - \mathbf{T}_2] &= (\mathbf{T}_1 - \mathbf{T}_2) \cdot (\mathbf{E}^c_2 - \mathbf{E}^c_1) + (\mathbf{T}_1 - \mathbf{T}_2) \cdot (\mathbf{E}^t_2 - \mathbf{E}^t_1) \\ &= (\mathbf{T}_1 - \sigma^t\mathbf{I}) \cdot \mathbf{E}^t_2 + (\mathbf{T}_2 - \sigma^t\mathbf{I}) \cdot \mathbf{E}^t_1 \\ &\quad + (\mathbf{T}_1 + \sigma^c\mathbf{I}) \cdot \mathbf{E}^c_2 + (\mathbf{T}_2 + \sigma^c\mathbf{I}) \cdot \mathbf{E}^c_1 \leqslant 0. \end{aligned}$$

Consequently, $\mathbf{T}_1 = \mathbf{T}_2$, $\mathbf{E}^e_1 = \mathbf{E}^e_2$ and $\mathbf{E}^t_1 + \mathbf{E}^c_1 = \mathbf{E}^t_2 + \mathbf{E}^c_2$. On the other hand, by using eqn (4) we obtain $0 \leqslant (\mathbf{E}^c_2 - \mathbf{E}^c_1) \cdot (\mathbf{E}^t_1 - \mathbf{E}^t_2) = \mathbf{E}^c_2 \cdot \mathbf{E}^t_1 + \mathbf{E}^c_1 \cdot \mathbf{E}^t_2 \leqslant 0$ and finally, $\mathbf{E}^c_2 - \mathbf{E}^c_1 = \mathbf{E}^t_1 - \mathbf{E}^t_2 = 0$.

By a procedure similar to that used by Lucchesi *et al.* (1994b), it is easy to prove that tensors $\mathbf{E}$, $\mathbf{E}^t$, $\mathbf{E}^c$, $\mathbf{E}^e$, $\mathbf{T}$, $\mathbf{T}+\sigma^c\mathbf{I}$ and $\mathbf{T}-\sigma^t\mathbf{I}$ are coaxial, so the constitutive equations (1)–(5) and (7)–(9) can be written with respect to the basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ of the eigenvectors of $\mathbf{E}$.

Let $\{e_1, e_2, e_3\}$, $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$ and $\{t_1, t_2, t_3\}$ be the eigenvalues of $\mathbf{E}$, $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$, respectively; the constitutive equations (1)–(5) and (7)–(9) are equivalent to the system:

$$
\begin{aligned}
t_1 &= 2\mu(e_1-a_1-b_1)+\lambda(e_1+e_2+e_3-a_1-a_2-a_3-b_1-b_2-b_3)\\
t_2 &= 2\mu(e_2-a_2-b_2)+\lambda(e_1+e_2+e_3-a_1-a_2-a_3-b_1-b_2-b_3)\\
t_3 &= 2\mu(e_3-a_3-b_3)+\lambda(e_1+e_2+e_3-a_1-a_2-a_3-b_1-b_2-b_3)\\
&(t_1-\sigma^t)a_1 = 0\\
&(t_2-\sigma^t)a_2 = 0\\
&(t_3-\sigma^t)a_3 = 0\\
&(t_1+\sigma^c)b_1 = 0\\
&(t_2+\sigma^c)b_2 = 0\\
&(t_3+\sigma^c)b_3 = 0\\
&a_1 \geqslant 0, \quad a_2 \geqslant 0, \quad a_3 \geqslant 0\\
&b_1 \leqslant 0, \quad b_2 \leqslant 0, \quad b_3 \leqslant 0\\
&t_1-\sigma^t \leqslant 0, \quad t_2-\sigma^t \leqslant 0, \quad t_3-\sigma^t \leqslant 0\\
&t_1+\sigma^c \geqslant 0, \quad t_2+\sigma^c \geqslant 0, \quad t_3+\sigma^c \geqslant 0\\
&a_1b_1+a_2b_2+a_3b_3 = 0.
\end{aligned}
\tag{10}
$$

Given the total deformation $\mathbf{E}$, the elastic moduli μ and λ and the material constants σ^c and σ^t, the principal components $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$ and $\{t_1, t_2, t_3\}$ satisfying eqn (10) can be calculated as functions of e_1, e_2, e_3. Their calculation requires definition of the following sets:

$$\mathscr{R}_1 = \{\mathbf{E}\in \mathrm{Sym}\,; 2e_3+\alpha\,\mathrm{tr}\,\mathbf{E}-\varepsilon^t \leqslant 0, \quad 2e_1+\alpha\,\mathrm{tr}\,\mathbf{E}+\varepsilon^c \geqslant 0\},$$

$$\mathscr{R}_2 = \{\mathbf{E}\in \mathrm{Sym}\,; 2e_1+\alpha\,\mathrm{tr}\,\mathbf{E}+\varepsilon^c < 0,$$
$$2\alpha e_2+4(1+\alpha)e_3-\alpha\varepsilon^c-(2+\alpha)\varepsilon^t \leqslant 0, \quad 2(1+\alpha)e_2+\alpha e_3+\varepsilon^c \geqslant 0\},$$

$$\mathscr{R}_3 = \{\mathbf{E}\in \mathrm{Sym}\,; 2(1+\alpha)e_2+\alpha e_3+\varepsilon^c < 0,$$
$$(2+3\alpha)e_3-\alpha\varepsilon^c-(1+\alpha)\varepsilon^t \leqslant 0, (2+3\alpha)e_3+\varepsilon^c \geqslant 0\},$$

$$\mathscr{R}_4 = \{\mathbf{E}\in \mathrm{Sym}\,; (2+3\alpha)e_3+\varepsilon^c < 0\},$$

$$\mathscr{R}_5 = \{\mathbf{E}\in \mathrm{Sym}\,; 2e_3+\alpha\,\mathrm{tr}\,\mathbf{E}-\varepsilon^t > 0, 2(1+\alpha)e_2+\alpha e_1-\varepsilon^t \leqslant 0,$$
$$4(1+\alpha)e_1+2\alpha e_3+\alpha\varepsilon^t+(2+\alpha)\varepsilon^c \geqslant 0\},$$

$$\mathscr{R}_6 = \{\mathbf{E}\in \mathrm{Sym}\,; 2(1+\alpha)e_2+\alpha e_1-\varepsilon^t > 0, (2+3\alpha)e_1-\varepsilon^t \leqslant 0,$$
$$(2+3\alpha)e_1+\alpha\varepsilon^t+(1+\alpha)\varepsilon^c \geqslant 0\},$$

$$\mathscr{R}_7 = \{\mathbf{E}\in \mathrm{Sym}\,; (2+3\alpha)e_1-\varepsilon^t \geqslant 0\},$$

$$\mathscr{R}_8 = \{\mathbf{E}\in \mathrm{Sym}\,; 2(2+3\alpha)e_2-\alpha\varepsilon^c-(2+\alpha)\varepsilon^t \geqslant 0, (2+3\alpha)e_1+\alpha\varepsilon^t+(1+\alpha)\varepsilon^c \leqslant 0\},$$

$$\mathscr{R}_9 = \{\mathbf{E}\in \mathrm{Sym}\,; 2(2+3\alpha)e_2+\alpha\varepsilon^t+(2+\alpha)\varepsilon^c \leqslant 0, (2+3\alpha)e_3-\alpha\varepsilon^c-(1+\alpha)\varepsilon^t \geqslant 0\},$$

$$\mathscr{R}_{10} = \{\mathbf{E}\in \mathrm{Sym}\,; 4(1+\alpha)e_1+2\alpha e_2+\alpha\varepsilon^t+(2+\alpha)\varepsilon^c < 0,$$
$$4(1+\alpha)e_3+2\alpha e_2-\alpha\varepsilon^c-(2+\alpha)\varepsilon^t > 0,$$
$$2(2+3\alpha)e_2-\alpha\varepsilon^c-(2+\alpha)\varepsilon^t \leqslant 0, 2(2+3\alpha)e_2+\alpha\varepsilon^t+(2+\alpha)\varepsilon^c \geqslant 0\},$$

where we have put $\alpha = \lambda/\mu$,† $\varepsilon^c = \sigma^c/\mu$ and $\varepsilon^t = \sigma^t/\mu$. Moreover, we suppose that the eigenvalues e_1, e_2 and e_3 are ordered in such a way that $e_1 \leqslant e_2 \leqslant e_3$. It is easy to prove that in the regions $\mathscr{R}_2$, $\mathscr{R}_6$ and $\mathscr{R}_8$ we have $e_1 < e_2 \leqslant e_3$ and that in $\mathscr{R}_3$, $\mathscr{R}_5$ and $\mathscr{R}_9$, $e_1 \leqslant e_2 < e_3$; finally in $\mathscr{R}_{10}$ the eigenvalues of $\mathbf{E}$ are distinct.

Solving system (10), we obtain the principal components of $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$:

$$\text{if } \mathbf{E} \in \mathscr{R}_1 \text{ then } \begin{aligned} a_1 &= 0,\\ a_2 &= 0,\\ a_3 &= 0,\\ b_1 &= 0,\\ b_2 &= 0,\\ b_3 &= 0,\\ t_1 &= \mu[(2+\alpha)e_1 + \alpha(e_2+e_3)],\\ t_2 &= \mu[(2+\alpha)e_2 + \alpha(e_1+e_3)],\\ t_3 &= \mu[(2+\alpha)e_3 + \alpha(e_1+e_2)]; \end{aligned} \tag{11}$$

$$\text{if } \mathbf{E} \in \mathscr{R}_2 \text{ then } \begin{aligned} a_1 &= 0,\\ a_2 &= 0,\\ a_3 &= 0,\\ b_1 &= e_1 + \frac{\alpha}{2+\alpha}(e_2+e_3) + \frac{\varepsilon^c}{2+\alpha},\\ b_2 &= 0,\\ b_3 &= 0,\\ t_1 &= -\sigma^c,\\ t_2 &= \mu\left\{2e_2 + \frac{\alpha}{2+\alpha}[2(e_2+e_3)-\varepsilon^c]\right\},\\ t_3 &= \mu\left\{2e_3 + \frac{\alpha}{2+\alpha}[2(e_2+e_3)-\varepsilon^c]\right\}; \end{aligned} \tag{12}$$

$$\text{if } \mathbf{E} \in \mathscr{R}_3 \text{ then } \begin{aligned} a_1 &= 0,\\ a_2 &= 0,\\ a_3 &= 0,\\ b_1 &= e_1 + \frac{\alpha}{2(1+\alpha)}e_3 + \frac{\varepsilon^c}{2(1+\alpha)},\\ b_2 &= e_2 + \frac{\alpha}{2(1+\alpha)}e_3 + \frac{\varepsilon^c}{2(1+\alpha)},\\ b_3 &= 0,\\ t_1 &= -\sigma^c,\\ t_2 &= -\sigma^c,\\ t_3 &= \frac{\mu}{1+\alpha}[(2+3\alpha)e_3 - \alpha\varepsilon^c]; \end{aligned} \tag{13}$$

† In the following we assume $\lambda \geqslant 0$, so that we have $\alpha \geqslant 0$.

$$
\text{if } \mathbf{E} \in \mathscr{R}_4 \text{ then } \quad
\begin{aligned}
a_1 &= 0,\\
a_2 &= 0,\\
a_3 &= 0,\\
b_1 &= e_1 + \frac{\varepsilon^c}{2+3\alpha},\\
b_2 &= e_2 + \frac{\varepsilon^c}{2+3\alpha},\\
b_3 &= e_3 + \frac{\varepsilon^c}{2+3\alpha},\\
t_1 &= -\sigma^c,\\
t_2 &= -\sigma^c,\\
t_3 &= -\sigma^c,
\end{aligned} \tag{14}
$$

$$
\text{if } \mathbf{E} \in \mathscr{R}_5 \text{ then } \quad
\begin{aligned}
a_1 &= 0,\\
a_2 &= 0,\\
a_3 &= e_3 + \frac{\alpha}{2+\alpha}(e_1 + e_2) - \frac{\varepsilon^t}{2+\alpha},\\
b_1 &= 0,\\
b_2 &= 0,\\
b_3 &= 0,\\
t_1 &= \frac{\mu}{2+\alpha}[4(1+\alpha)e_1 + 2\alpha e_2 + \alpha\varepsilon^t],\\
t_2 &= \frac{\mu}{2+\alpha}[4(1+\alpha)e_2 + 2\alpha e_1 + \alpha\varepsilon^t],\\
t_3 &= \sigma^t;
\end{aligned} \tag{15}
$$

$$
\text{if } \mathbf{E} \in \mathscr{R}_6 \text{ then } \quad
\begin{aligned}
a_1 &= 0,\\
a_2 &= e_2 + \frac{\alpha}{2(1+\alpha)} e_1 - \frac{\varepsilon^t}{2(1+\alpha)},\\
a_3 &= e_3 + \frac{\alpha}{2(1+\alpha)} e_1 - \frac{\varepsilon^t}{2(1+\alpha)},\\
b_1 &= 0,\\
b_2 &= 0,\\
b_3 &= 0,\\
t_1 &= \frac{\mu}{1+\alpha}[(2+3\alpha)e_1 + \alpha\varepsilon^t],\\
t_2 &= \sigma^t,\\
t_3 &= \sigma^t;
\end{aligned} \tag{16}
$$

$$
\text{if } \mathbf{E}\in\mathscr{R}_7 \text{ then}\quad
\begin{aligned}
a_1 &= e_1-\frac{\varepsilon^{t}}{2+3\alpha},\\
a_2 &= e_2-\frac{\varepsilon^{t}}{2+3\alpha},\\
a_3 &= e_3-\frac{\varepsilon^{t}}{2+3\alpha},\\
b_1 &= 0,\\
b_2 &= 0,\\
b_3 &= 0,\\
t_1 &= \sigma^{t},\\
t_2 &= \sigma^{t},\\
t_3 &= \sigma^{t};
\end{aligned}
\tag{17}
$$

$$
\text{if } \mathbf{E}\in\mathscr{R}_8 \text{ then}\quad
\begin{aligned}
a_1 &= 0,\\
a_2 &= e_2-\frac{\alpha\varepsilon^{c}}{2(2+3\alpha)}-\frac{(2+\alpha)\varepsilon^{t}}{2(2+3\alpha)},\\
a_3 &= e_3-\frac{\alpha\varepsilon^{c}}{2(2+3\alpha)}-\frac{(2+\alpha)\varepsilon^{t}}{2(2+3\alpha)},\\
b_1 &= e_1+\frac{(\alpha+1)\varepsilon^{c}}{2+3\alpha}+\frac{\alpha\varepsilon^{t}}{2+3\alpha},\\
b_2 &= 0,\\
b_3 &= 0,\\
t_1 &= -\sigma^{c},\\
t_2 &= \sigma^{t},\\
t_3 &= \sigma^{t};
\end{aligned}
\tag{18}
$$

$$
\text{if } \mathbf{E}\in\mathscr{R}_9 \text{ then}\quad
\begin{aligned}
a_1 &= 0,\\
a_2 &= 0,\\
a_3 &= e_3-\frac{\alpha\varepsilon^{c}}{2+3\alpha}-\frac{(1+\alpha)\varepsilon^{t}}{2+3\alpha},\\
b_1 &= e_1+\frac{(\alpha+2)\varepsilon^{c}}{2(2+3\alpha)}+\frac{\alpha\varepsilon^{t}}{2(2+3\alpha)},\\
b_2 &= e_2+\frac{(\alpha+2)\varepsilon^{c}}{2(2+3\alpha)}+\frac{\alpha\varepsilon^{t}}{2(2+3\alpha)}\\
b_3 &= 0,\\
t_1 &= -\sigma^{c},\\
t_2 &= -\sigma^{c},\\
t_3 &= \sigma^{t};
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\text{if } \mathbf{E}\in\mathcal{R}_{10} \text{ then } \quad a_1 &= 0,\\
a_2 &= 0,\\
a_3 &= e_3+\frac{\alpha}{2(1+\alpha)}e_2-\frac{\alpha+2}{4(1+\alpha)}\varepsilon^{\mathrm{t}}-\frac{\alpha}{4(1+\alpha)}\varepsilon^{\mathrm{c}},\\
b_1 &= e_1+\frac{\alpha}{2(1+\alpha)}e_2+\frac{\alpha+2}{4(1+\alpha)}\varepsilon^{\mathrm{c}}+\frac{\alpha}{4(1+\alpha)}\varepsilon^{\mathrm{t}},\\
b_2 &= 0,\\
b_3 &= 0,\\
t_1 &= -\sigma^{\mathrm{c}},\\
t_2 &= \frac{\mu}{2(1+\alpha)}[2(2+3\alpha)e_2+\alpha(\varepsilon^{\mathrm{t}}-\varepsilon^{\mathrm{c}})],\\
t_3 &= \sigma^{\mathrm{t}}.
\end{aligned} \tag{20}
$$

Therefore, given a symmetric tensor $\mathbf{E}=\Sigma_{i=1}^{3}e_i\mathbf{q}_i\otimes\mathbf{q}_i$ and having determined the region $\mathcal{R}_k$ to which $\mathbf{E}$ belongs, the solution to the constitutive equations (1)–(5) and (7)–(9) is

$$
\mathbf{E}^{\mathrm{t}}=\sum_{i=1}^{3}a_i\mathbf{q}_i\otimes\mathbf{q}_i,\quad \mathbf{E}^{\mathrm{c}}=\sum_{i=1}^{3}b_i\mathbf{q}_i\otimes\mathbf{q}_i,\quad \mathbf{T}=\sum_{i=1}^{3}t_i\mathbf{q}_i\otimes\mathbf{q}_i,
$$

with a_i, b_i and t_i given in eqns (11)–(20).

We shall denote by $\hat{\mathbf{T}}$ the function $\hat{\mathbf{T}}$: Sym → Sym which associates to every tensor $\mathbf{E}=\Sigma_{i=1}^{3}e_i\mathbf{q}_i\otimes\mathbf{q}_i$ the stress $\mathbf{T}=\hat{\mathbf{T}}(\mathbf{E})=\Sigma_{i=1}^{3}t_i\mathbf{q}_i\otimes\mathbf{q}_i$. $\hat{\mathbf{T}}$ is a continuous non-linear, non-injective function, positively homogeneous of degree one (Del Piero, 1989),

$$
\hat{\mathbf{T}}(\alpha\mathbf{E})=\alpha\hat{\mathbf{T}}(\mathbf{E})\quad\forall\,\alpha\geqslant 0,\quad\forall\,\mathbf{E}\in\mathrm{Sym}
$$

and isotropic,

$$
\hat{\mathbf{T}}(\mathbf{QEQ}^{\mathrm{T}})=\mathbf{Q}\hat{\mathbf{T}}(\mathbf{E})\mathbf{Q}^{\mathrm{T}}\quad\forall\,\mathbf{Q}\in\mathrm{Orth},\forall\,\mathbf{E}\in\mathrm{Sym}\,;\dagger
$$

moreover, we shall prove that $\hat{\mathbf{T}}$ is differentiable in the internal part of every region $\mathcal{R}_i$.

Now we analyse the plane strain and the plane stress separately. If $\mathbf{E}$ is a plane strain and, in particular, $e_3=\mathbf{q}_3\cdot\mathbf{E}\,\mathbf{q}_3=0$, then $a_3=b_3=0$ and $t_3=[\alpha/2(1+\alpha)]\,(t_1+t_2)$. Let us designate $\mathbf{E}$, $\mathbf{E}^{\mathrm{t}}$, $\mathbf{E}^{\mathrm{c}}$ and $\mathbf{T}$ as the restrictions of $\mathbf{E}$, $\mathbf{E}^{\mathrm{t}}$, $\mathbf{E}^{\mathrm{c}}$ and $\mathbf{T}$ to the two-dimensional subspace of $\mathscr{V}$, orthogonal to the vector $\mathbf{q}_3$. Calculation of a_1, a_2, b_1, b_2, t_1 and t_2 which satisfy system (10) requires definition of the following sets:

$$
\begin{aligned}
\mathscr{S}_1 &= \{\mathbf{E}\in\mathrm{Sym}\,;\alpha e_1+(2+\alpha)e_2-\varepsilon^{\mathrm{t}}\leqslant 0,(2+\alpha)e_1+\alpha e_2+\varepsilon^{\mathrm{c}}\geqslant 0\},\\
\mathscr{S}_2 &= \left\{\mathbf{E}\in\mathrm{Sym}\,;e_1>\frac{\varepsilon^{\mathrm{t}}}{2(1+\alpha)}\right\},\\
\mathscr{S}_3 &= \left\{\mathbf{E}\in\mathrm{Sym}\,;\alpha e_1+(2+\alpha)e_2-\varepsilon^{\mathrm{t}}>0,\quad e_1\leqslant\frac{\varepsilon^{\mathrm{t}}}{2(1+\alpha)},\quad e_1\geqslant-\frac{(2+\alpha)\varepsilon^{\mathrm{c}}+\alpha\varepsilon^{\mathrm{t}}}{4(1+\alpha)}\right\},\\
\mathscr{S}_4 &= \left\{\mathbf{E}\in\mathrm{Sym}\,;(2+\alpha)e_1+\alpha e_2+\varepsilon^{\mathrm{c}}<0,\quad e_2\geqslant-\frac{\varepsilon^{\mathrm{c}}}{2(1+\alpha)},\quad e_2\leqslant\frac{\alpha\varepsilon^{\mathrm{c}}+(2+\alpha)\varepsilon^{\mathrm{t}}}{4(1+\alpha)}\right\},
\end{aligned}
$$

† Orth denotes the set of all tensors $\mathbf{Q}$ such that $\mathbf{Q}^{\mathrm{T}}=\mathbf{Q}^{-1}$.

$$\mathcal{S}_5 = \left\{ \mathbf{E} \in \mathrm{Sym}\, ; e_2 < -\frac{\varepsilon^c}{2(1+\alpha)} \right\},$$

$$\mathcal{S}_6 = \left\{ \mathbf{E} \in \mathrm{Sym}\, ; e_2 > \frac{\alpha\varepsilon^c + (2+\alpha)\varepsilon^t}{4(1+\alpha)}, \quad e_1 < -\frac{\alpha\varepsilon^t + (2+\alpha)\varepsilon^c}{4(1+\alpha)} \right\}.$$

We still suppose that the eigenvalues e_1 and e_2 of $\mathbf{E}$ are ordered in such a way that $e_1 \leqslant e_2$. We observe that in $\mathcal{S}_3$, $\mathcal{S}_4$ and $\mathcal{S}_6$ the eigenvalues e_1 and e_2 are distinct. Regions $\mathcal{S}_1, \ldots, \mathcal{S}_6$ in the e_1–e_2 plane are illustrated in Fig. 1.

The principal components of $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$ can be calculated from the relations

$$\text{if } \mathbf{E} \in \mathcal{S}_1, \text{ then } \quad \begin{aligned} a_1 &= 0, & a_2 &= 0, \\ b_1 &= 0, & b_2 &= 0, \\ t_1 &= \mu[(2+\alpha)e_1 + \alpha e_2], & t_2 &= \mu[(2+\alpha)e_2 + \alpha e_1]; \end{aligned} \tag{21}$$

$$\text{if } \mathbf{E} \in \mathcal{S}_2, \text{ then } \quad \begin{aligned} a_1 &= e_1 - \frac{\varepsilon^t}{2(1+\alpha)}, & a_2 &= e_2 - \frac{\varepsilon^t}{2(1+\alpha)}, \\ b_1 &= 0, & b_2 &= 0, \\ t_1 &= \sigma^t, & t_2 &= \sigma^t; \end{aligned} \tag{22}$$

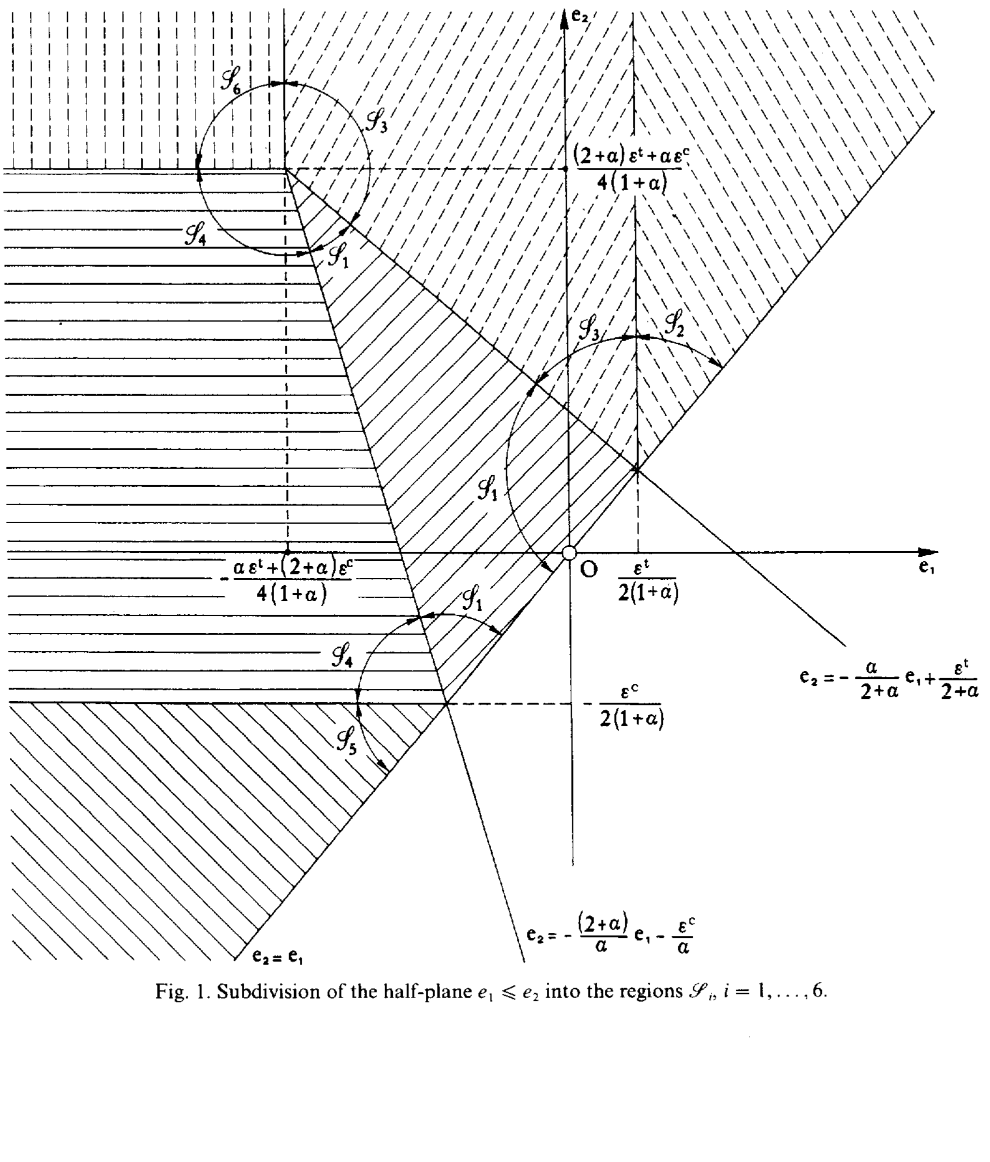

Fig. 1. Subdivision of the half-plane $e_1 \leqslant e_2$ into the regions $\mathcal{S}_i$, $i = 1, \ldots, 6$.

$$\text{if } \mathbf{E}\in\mathscr{S}_3, \text{ then } \quad a_1 = 0, \qquad a_2 = e_2 + \frac{\alpha}{2+\alpha}e_1 - \frac{\varepsilon^t}{2+\alpha}$$

$$b_1 = 0, \qquad b_2 = 0,$$

$$t_1 = \frac{4\mu(1+\alpha)}{2+\alpha}e_1 + \frac{\alpha}{2+\alpha}\sigma^t, \quad t_2 = \sigma^t; \tag{23}$$

$$\text{if } \mathbf{E}\in\mathscr{S}_4, \text{ then } \quad a_1 = 0, \qquad a_2 = 0,$$

$$b_1 = e_1 + \frac{\alpha}{2+\alpha}e_2 + \frac{\varepsilon^c}{2+\alpha}, \quad b_2 = 0,$$

$$t_1 = -\sigma^c, \qquad t_2 = \frac{4\mu(1+\alpha)}{2+\alpha}e_2 - \frac{\alpha}{2+\alpha}\sigma^c; \tag{24}$$

$$\text{if } \mathbf{E}\in\mathscr{S}_5, \text{ then } \quad a_1 = 0, \qquad a_2 = 0,$$

$$b_1 = e_1 + \frac{\varepsilon^c}{2(1+\alpha)}, \quad b_2 = e_2 + \frac{\varepsilon^c}{2(1+\alpha)},$$

$$t_1 = -\sigma^c, \qquad t_2 = -\sigma^c, \tag{25}$$

$$\text{if } \mathbf{E}\in\mathscr{S}_6, \text{ then } \quad a_1 = 0, \qquad a_2 = e_2 - \frac{\alpha\varepsilon^c + (2+\alpha)\varepsilon^t}{4(1+\alpha)},$$

$$b_1 = e_1 + \frac{(2+\alpha)\varepsilon^c + \alpha\varepsilon^t}{4(1+\alpha)}, \quad b_2 = 0,$$

$$t_1 = -\sigma^c, \qquad t_2 = \sigma^t. \tag{26}$$

From the relation $t_3 = [\alpha/2(1+\alpha)](t_1+t_2)$ and from the non-negativeness of α, it follows that the eigenvalue t_3 of $\mathbf{T}$ satisfies the inequalities $-\sigma^c \leqslant t_3 \leqslant \sigma^t$ as well.

Now let us consider a plane stress and suppose $t_3 = \mathbf{q}_3 \cdot \mathbf{T}\mathbf{q}_3 = 0$. Then a_3 can be set equal to zero and b_3, by virtue of the positiveness of σ^c, must be equal to zero, so that we have $e_3 = [\alpha/2+\alpha](a_1+a_2+b_1+b_2-e_1-e_2)$. Let us still designate $\mathbf{E}$, $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$ as the restrictions of $\mathbf{E}$, $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$ to the two-dimensional subspace of $\mathscr{V}$, orthogonal to the vector $\mathbf{q}_3$. In order to calculate the values of a_1, a_2, b_1, b_2, t_1 and t_2 which satisfy system (10) we define the following sets:

$$\mathscr{T}_1 = \{\mathbf{E}\in\text{Sym}\,; 2\alpha e_1 + 4(1+\alpha)e_2 - \varepsilon^t(2+\alpha) \leqslant 0, 4(1+\alpha)e_1 + 2\alpha e_2 + \varepsilon^c(2+\alpha) \geqslant 0\},$$

$$\mathscr{T}_2 = \left\{\mathbf{E}\in\text{Sym}\,; e_1 > \frac{(2+\alpha)\varepsilon^t}{2(2+3\alpha)}\right\},$$

$$\mathscr{T}_3 = \left\{\mathbf{E}\in\text{Sym}\,; 2\alpha e_1 + 4(1+\alpha)e_2 - (2+\alpha)\varepsilon^t > 0, e_1 \leqslant \frac{(2+\alpha)\varepsilon^t}{2(2+3\alpha)}, e_1 \geqslant -\frac{2(1+\alpha)\varepsilon^c + \alpha\varepsilon^t}{2(2+3\alpha)}\right\},$$

$$\mathscr{T}_4 = \left\{\mathbf{E}\in\text{Sym}\,; 4(1+\alpha)e_1 + 2\alpha e_2 + (2+\alpha)\varepsilon^c < 0, e_2 \geqslant -\frac{(2+\alpha)\varepsilon^c}{2(2+3\alpha)}, e_2 \leqslant \frac{\alpha\varepsilon^c + 2(1+\alpha)\varepsilon^t}{2(2+3\alpha)}\right\},$$

$$\mathscr{T}_5 = \left\{\mathbf{E}\in\text{Sym}\,; e_2 < -\frac{(2+\alpha)\varepsilon^c}{2(2+3\alpha)}\right\},$$

$$\mathscr{T}_6 = \left\{\mathbf{E}\in\text{Sym}\,; e_2 > \frac{\alpha\varepsilon^c + 2(1+\alpha)\varepsilon^t}{2(2+3\alpha)}, e_1 < -\frac{\alpha\varepsilon^t + 2(1+\alpha)\varepsilon^c}{2(2+3\alpha)}\right\}.$$

We observe that in $\mathscr{T}_3$, $\mathscr{T}_4$ and $\mathscr{T}_6$ the eigenvalues e_1 and e_2 are distinct. The principal components of $\mathbf{E}^t$, $\mathbf{E}^c$ and $\mathbf{T}$ can be calculated from the relations:

$$\text{if } \mathbf{E}\in\mathscr{T}_1, \text{ then } \quad a_1 = 0, \quad a_2 = 0,$$
$$b_1 = 0, \quad b_2 = 0,$$
$$t_1 = 2\mu\left[e_1 + \frac{\alpha}{2+\alpha}(e_1+e_2)\right], \quad t_2 = 2\mu\left[e_2 + \frac{\alpha}{2+\alpha}(e_1+e_2)\right]; \tag{27}$$

$$\text{if } \mathbf{E}\in\mathscr{T}_2, \text{ then } \quad a_1 = e_1 - \frac{(2+\alpha)}{2(2+3\alpha)}\varepsilon^t, \quad a_2 = e_2 - \frac{(2+\alpha)}{2(2+3\alpha)}\varepsilon^t,$$
$$b_1 = 0, \quad b_2 = 0,$$
$$t_1 = \sigma^t, \quad t_2 = \sigma^t; \tag{28}$$

$$\text{if } \mathbf{E}\in\mathscr{T}_3, \text{ then } \quad a_1 = 0, \quad a_2 = e_2 + \frac{\alpha}{2(1+\alpha)}e_1 - \frac{(2+\alpha)}{4(1+\alpha)}\varepsilon^t,$$
$$b_1 = 0, \quad b_2 = 0,$$
$$t_1 = \frac{\mu(2+3\alpha)}{1+\alpha}e_1 + \frac{\alpha}{2(1+\alpha)}\sigma^t, t_2 = \sigma^t; \tag{29}$$

$$\text{if } \mathbf{E}\in\mathscr{T}_4, \text{ then } \quad a_1 = 0, \quad a_2 = 0,$$
$$b_1 = e_1 + \frac{\alpha}{2(1+\alpha)}e_2 + \frac{(2+\alpha)\varepsilon^c}{4(1+\alpha)}, \quad b_2 = 0,$$
$$t_1 = -\sigma^c, \quad t_2 = \frac{\mu(2+3\alpha)}{1+\alpha}e_2 - \frac{\alpha}{2(1+\alpha)}\sigma^c; \tag{30}$$

$$\text{if } \mathbf{E}\in\mathscr{T}_5, \text{ then } \quad a_1 = 0, \quad a_2 = 0,$$
$$b_1 = e_1 + \frac{(2+\alpha)\varepsilon^c}{2(2+3\alpha)}, \quad b_2 = e_2 + \frac{(2+\alpha)\varepsilon^c}{2(2+3\alpha)},$$
$$t_1 = -\sigma^c, \quad t_2 = -\sigma^c; \tag{31}$$

$$\text{if } \mathbf{E}\in\mathscr{T}_6, \text{ then } \quad a_1 = 0, \quad a_2 = e_2 - \frac{\alpha\varepsilon^c + 2(1+\alpha)\varepsilon^t}{2(2+3\alpha)},$$
$$b_1 = e_1 + \frac{2(1+\alpha)\varepsilon^c + \alpha\varepsilon^t}{2(2+3\alpha)}, \quad b_2 = 0,$$
$$t_1 = -\sigma^c, \quad t_2 = \sigma^t. \tag{32}$$

3. THE BOUNDARY-VALUE PROBLEM

The equilibrium problem for masonry-like solids (infinitely resistant to compression and with $\sigma^t = 0$) has been studied in recent years and the existence of a solution has been proven solely for a rather restricted class of load conditions (Giaquinta and Giusti, 1985; Anzellotti, 1985). On the other hand, the uniqueness of the solution is guaranteed only in terms of stress, in the sense that different displacement and strain fields can correspond to the same stress field.

Similar considerations can be made for a BCS masonry-like material; in this section we prove that the stress field which satisfies the equilibrium problem for a BCS masonry-like material is unique. To this end, let $\mathscr{B}$ be a solid made up of a BCS material and let $\mathscr{S}_u$ and $\mathscr{S}_f$ be two subsets of the boundary $\partial\mathscr{B}$ of $\mathscr{B}$, such that their union covers $\partial\mathscr{B}$ and their interiors are disjointed.

A load $(\mathbf{b}, \mathbf{s}_0)$ defined in $\mathscr{B} \times \mathscr{S}_f$ with values in $\mathscr{V} \times \mathscr{V}$ is *admissible* if the corresponding boundary-value problem has a solution, i.e. if there exists a triple $[\mathbf{u}, \mathbf{E}, \mathbf{T}]$, constituted by a stress field $\mathbf{T}$, a strain field $\mathbf{E}$ and a displacement field $\mathbf{u}$ defined on $\bar{\mathscr{B}}$, piecewise C^2, such that

$$\mathbf{E} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \tag{33}$$

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E}) = \mathbb{C}[\mathbf{E} - \mathbf{E}^t - \mathbf{E}^c], \tag{34}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \mathscr{S}_u, \tag{35}$$

$$\mathbf{T}\mathbf{n} = \mathbf{s}_0 \quad \text{on} \quad \mathscr{S}_f, \tag{36}$$

$$\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathscr{B}, \tag{37}$$

where $\mathbf{n}$ is the outward unit normal to $\mathscr{S}_f$, $\mathbb{C} = 2\mu\,\mathbb{1} + \lambda\mathbf{I}\otimes\mathbf{I}$ is the elasticity tensor and $\mathbf{E}^t$ and $\mathbf{E}^c$ satisfy with $\mathbf{T}$ the constitutive equations (1)–(9). It is easy to prove that if $(\mathbf{b}, \mathbf{s}_0)$ is an admissible load and $[\mathbf{u}_1, \mathbf{E}_1, \mathbf{T}_1]$ and $[\mathbf{u}_2, \mathbf{E}_2, \mathbf{T}_2]$ are two solutions to eqns (33)–(37), then $\mathbf{T}_1(x) = \mathbf{T}_2(x)$ for every $x \in \mathscr{B}$.

In fact, the triple $[\bar{\mathbf{u}}, \bar{\mathbf{E}}, \bar{\mathbf{T}}]$ with $\bar{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$, $\bar{\mathbf{E}} = \mathbf{E}_1 - \mathbf{E}_2$ and $\bar{\mathbf{T}} = \mathbf{T}_1 - \mathbf{T}_2$ satisfies eqns (33) and (35); moreover it satisfies eqns (36) and (37) with $\mathbf{s}_0 = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$. Thus, in agreement with the hypothesis on the smoothness of the solutions, a simple application of the principle of virtual work proves that:

$$\int_{\mathscr{B}} \bar{\mathbf{T}} \cdot \bar{\mathbf{E}}\,\mathrm{d}V = 0. \tag{38}$$

On the other hand,

$$\bar{\mathbf{E}} = \bar{\mathbf{E}}^e + \mathbf{E}_1^t + \mathbf{E}_1^c - \mathbf{E}_2^t - \mathbf{E}_2^c, \tag{39}$$

where $\bar{\mathbf{E}}^e = \mathbf{E}_1^e - \mathbf{E}_2^e$, and $\mathbf{E}_1^e, \mathbf{E}_1^t, \mathbf{E}_1^c, \mathbf{E}_2^e, \mathbf{E}_2^t$ and $\mathbf{E}_2^c$ are the elastic part, the fracture strain and the crushing strain corresponding to $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively. From eqn (38), by using eqn (39) we obtain:

$$\int_{\mathscr{B}} \bar{\mathbf{T}} \cdot \bar{\mathbf{E}}^e\,\mathrm{d}V = \int_{\mathscr{B}} (\mathbf{T}_1 - \mathbf{T}_2)\cdot(\mathbf{E}_2^t + \mathbf{E}_2^c - \mathbf{E}_1^t - \mathbf{E}_1^c)\,\mathrm{d}V; \tag{40}$$

the first member in eqn (40) is equal to $\int_{\mathscr{B}} \bar{\mathbf{T}} \cdot \mathbb{C}^{-1}[\bar{\mathbf{T}}]\,\mathrm{d}V$ and then it is non-negative because $\mathbb{C}$ is positive definite. By using eqn (9), the second member of eqn (40) results in:

$$\int_{\mathscr{B}} [(\mathbf{T}_1 - \sigma^t\mathbf{I})\cdot\mathbf{E}_2^t + (\mathbf{T}_2 - \sigma^t\mathbf{I})\cdot\mathbf{E}_1^t + (\mathbf{T}_1 + \sigma^c\mathbf{I})\cdot\mathbf{E}_2^c + (\mathbf{T}_2 + \sigma^c\mathbf{I})\cdot\mathbf{E}_1^c]\,\mathrm{d}V,$$

which is non-positive by virtue of eqns (2), (3), (7) and (8). From the equality (40) we obtain $\bar{\mathbf{T}}\cdot\mathbb{C}^{-1}[\bar{\mathbf{T}}] = 0$ everywhere in $\mathscr{B}$ and thus $\bar{\mathbf{T}} = 0$, which is the desired result.

In order to solve the equilibrium problems for BCS masonry-like solids by using the finite element method, we are often obliged for numerical reasons, to assign the load incrementally. To this end, although the material being considered is elastic, we must also consider the load processes and incremental equilibrium problem associated with them.

We then intend to prove that the numerical solution obtained by using an incremental procedure is independent of the particular load process chosen; instead, it depends solely on the final assigned load, provided that the load process considered is admissible in the sense specified as follows.

A load process $\gamma(\tau)$, $\tau \in [0, \bar{\tau}]$, is a function pair $[\mathbf{b}(x, \tau), s_0(x, \tau)]$ with $\mathbf{b}$ and $\mathbf{s}_0$ defined on $\mathscr{B} \times [0, \bar{\tau}]$ and $\mathscr{S}_f \times [0, \bar{\tau}]$, respectively, differentiable with respect to τ and such that $\gamma(0) = 0$. Given a process γ, let us suppose that for every τ, $\gamma(\tau) = [\mathbf{b}(x, \tau), \mathbf{s}_0(x, \tau)]$ is an admissible load and let $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ be a solution to eqns (33)–(37) with $\mathbf{b} = \mathbf{b}(\tau)$ and $\mathbf{s}_0 = \mathbf{s}_0(\tau)$. A curve $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ of solutions to eqns (33)–(37) is said to be *regular* if it is differentiable with respect to τ. A load process γ on $[0, \bar{\tau}]$ is *admissible* if, for every $\tau \in [0, \bar{\tau}]$, $\gamma(\tau)$ is an admissible load and if there exists at least one regular curve $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ of solutions to eqns (33)–(37). Let γ be a load process on $[0, \bar{\tau}]$; a regular curve $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ is an *incremental solution* to the boundary-value problem if for each $\tau \in [0, \bar{\tau}]$ we have

$$
\begin{aligned}
\dot{\mathbf{E}} &= \tfrac{1}{2}(\nabla\dot{\mathbf{u}} + \nabla\dot{\mathbf{u}}^t), \\
\dot{\mathbf{T}} &= D_E\hat{\mathbf{T}}(\mathbf{E}(\tau))[\dot{\mathbf{E}}], \\
\dot{\mathbf{u}} &= \mathbf{0} \quad \text{on} \quad \mathscr{S}_u, \\
\dot{\mathbf{T}}\mathbf{n} &= \dot{\mathbf{s}}_0 \quad \text{on} \quad \mathscr{S}_f, \\
\operatorname{div}\dot{\mathbf{T}} + \dot{\mathbf{b}} &= \mathbf{0} \quad \text{on} \quad \mathscr{B},
\end{aligned}
\tag{41}
$$

and

$$
\mathbf{u}(x, 0) = \mathbf{0}, \quad \mathbf{E}(x, 0) = \mathbf{0}, \quad \mathbf{T}(x, 0) = \mathbf{0} \quad \text{on} \quad \mathscr{B}, \tag{42}
$$

where the dot $\cdot$ denotes the derivatives with respect to τ.

It is immediately verifiable that, if γ is an admissible process, then every regular curve of solutions to eqns (33)–(37) is a solution to eqn (41). Moreover, each incremental solution to the boundary-value problem is a regular curve of solutions to eqns (33)–(37). In fact, if $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ is a regular curve of solutions to these equations, differentiating them with respect to τ, we can immediately verify that $[\mathbf{u}, \mathbf{E}, \mathbf{T}]$ satisfies eqn (41). On the other hand, if $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ is an incremental solution, integrating eqn (41) on $[0, \tau]$ and taking into account eqn (42), we deduce that $[\mathbf{u}(\tau), \mathbf{E}(\tau), \mathbf{T}(\tau)]$ satisfies eqns (33)–(37) for each $\tau \in [0, \bar{\tau}]$.

From this result it follows that:

(a) if γ is an admissible process, there exists at least one incremental solution to the boundary-value problem;
(b) the solution to the incremental problem, if it exists, is unique in terms of stress, i.e. if $[\mathbf{u}_1(\tau), \mathbf{E}_1(\tau), \mathbf{T}_1(\tau)]$ and $[\mathbf{u}_2(\tau), \mathbf{E}_2(\tau), \mathbf{T}_2(\tau)]$ are two solutions to eqn (41) then

$$
\mathbf{T}_1(x, \tau) = \mathbf{T}_2(x, \tau), \quad (x, \tau) \in \mathscr{B} \times [0, \bar{\tau}]. \tag{43}
$$

(c) if γ and φ are two admissible processes on $[0, \bar{\tau}]$, such that $\gamma(\bar{\tau}) = \varphi(\bar{\tau})$ and $[\mathbf{u}_1(\tau), \mathbf{E}_1(\tau), \mathbf{T}_1(\tau)]$ and $[\mathbf{u}_2(\tau), \mathbf{E}_2(\tau), \mathbf{T}_2(\tau)]$ are two incremental solutions corresponding to γ and φ, respectively, then

$$
\mathbf{T}_1(x, \bar{\tau}) = \mathbf{T}_2(x, \bar{\tau}) \quad \text{for each} \quad x \in \mathscr{B}. \tag{44}
$$

This last result guarantees that the incremental solution does not depend on the load process at least regarding the stress. In fact, the common value of $\mathbf{T}_1$ and $\mathbf{T}_2$ at the end of the two processes is the solution to the boundary-value problem [(33)–(37)] corresponding to the load $\gamma(\bar{\tau}) = \varphi(\bar{\tau})$.

4. SOME EXPLICIT SOLUTIONS

In this section we analyse a circular ring and a spherical container made up of a BCS material subjected to uniform radial pressures p_e and p_i acting, respectively, on the outer and inner boundary and we explicitly calculate the stress field at equilibrium with these loads and the corresponding strain and displacement fields that, in this case, are unique. The explicit solutions thus obtained will be compared in Section 6 with the corresponding numerical results.

In the following, ν and E are, respectively, the Poisson ratio and the Young modulus of the material. Moreover, we suppose $\sigma^t = 0$, $\sigma^c > 0$ to be fixed, and that p_e and p_i satisfy the compatibility conditions $p_e \leqslant \sigma^c$ and $p_i \leqslant \sigma^c$. A stress field in equilibrium with loads p_e and p_i, satisfying eqns (7) and (8) will be said to be *statically admissible*.

4.1. *The circular ring*

The circular ring Ω shown in Fig. 2, having inner radius a and outer radius b, is subjected to a plane strain as a consequence of the action of two uniform radial pressures p_e and p_i acting, respectively, on the outer and inner boundary. Let us choose a cylindrical reference system $\{O, \rho, \theta, z\}$ in which the origin coincides with the centre of the ring and the z axis is orthogonal to its plane.

It is known (Bennati and Padovani, 1992) that if $(p_e/p_i) \geqslant (a^2+b^2)/2b^2$, then the stress field $\mathbf{T}^{(e)}$ corresponding to a linear elastic material, having principal components

$$\sigma_\rho^{(e)}(\rho) = \frac{a^2b^2(p_e-p_i)}{b^2-a^2}\frac{1}{\rho^2}+\frac{p_ia^2-p_eb^2}{b^2-a^2},$$

$$\sigma_\theta^{(e)}(\rho) = -\frac{a^2b^2(p_e-p_i)}{b^2-a^2}\frac{1}{\rho^2}+\frac{p_ia^2-p_eb^2}{b^2-a^2},$$

$$\sigma_z^{(e)}(\rho) = \nu[\sigma_\rho^{(e)}(\rho)+\sigma_\theta^{(e)}(\rho)] = \frac{2\nu(p_ia^2-p_eb^2)}{b^2-a^2}, \tag{45}$$

is negative semi-definite.

Let us begin by supposing

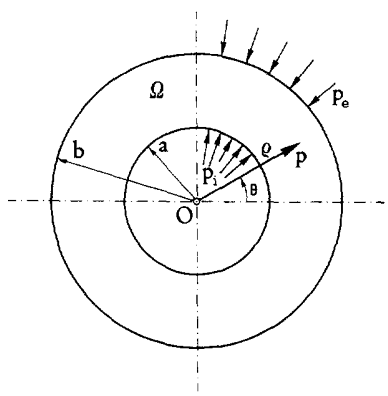

Fig. 2. The circular ring.

$$\frac{p_e}{p_i} \geqslant 1,$$

then, for the circumferential stress, which is a monotonic function of ρ, the inequalities $\sigma_\theta^{(e)}(a) \leqslant \sigma_\theta^{(e)}(b) \leqslant -p_e$ hold.

If, in particular p_e and p_i are such that the condition:

$$\frac{\sigma^c - p_e}{\sigma^c - p_i} \geqslant \frac{a^2+b^2}{2b^2}$$

is also satisfied, or, equivalently, if $p_e \leqslant [(a^2+b^2)/2b^2]p_i + [(b^2-a^2)/2b^2]\sigma^c$, then the stress field $\mathbf{T}^{(e)}$ is statically admissible. On the other hand, if p_e and p_i are such that the inequality:

$$\frac{\sigma^c - p_e}{\sigma^c - p_i} \leqslant \frac{a^2+b^2}{2b^2}$$

holds, then $\sigma_\theta^{(e)}(a) < -\sigma^c$, and $\mathbf{T}^{(e)}$ does not satisfy condition (8). A statically admissible stress field $\mathbf{T}$ can be obtained by starting from $\mathbf{T}^{(e)}$ and using a procedure similar to that by Bennati and Padovani (1992) for a circular ring made up of an elastic non-linear material with bounded tensile strength.

In the attempt to find the solution, we may suppose that $\sigma_\theta(\rho)$ is equal to $-\sigma^c$ in a circular ring $\Omega_1 = \{(\rho,\theta);\ \rho \in [a,\rho_c]\}$, where $\rho_c \in [a,b]$ is unknown. In this region, for equilibrium reasons, σ_ρ has the expression

$$\sigma_\rho(\rho) = \frac{a}{\rho}(\sigma^c - p_i) - \sigma^c.$$

Consequently, the circular ring $\Omega_2 = \{(\rho,\theta);\ \rho \in [\rho_c, b]\}$ is subjected to both external pressure p_e and an internal pressure whose value is $p_c = \sigma^c - (a/\rho_c)(\sigma^c - p_i)$. Moreover, for continuity reasons, $\sigma_\theta(\rho_c^+) = -\sigma^c$. On the other hand, in Ω_2 the solution coincides with the linear elastic one; thus,

$$\sigma_\theta(\rho_c^+) = \frac{p_c(b^2+\rho_c^2) - 2p_e b^2}{b^2 - \rho_c^2}$$

and ρ_c is a solution to the algebraic equation

$$a(\sigma^c - p_i)\rho_c^2 - 2b^2(\sigma^c - p_e)\rho_c + ab^2(\sigma^c - p_i) = 0,$$

which, if $(\sigma^c - p_e)/(\sigma^c - p_i) \geqslant (a/b)$, that is if $p_e \leqslant (a/b)p_i + [(b-a)/b]\sigma^c$, has in $[a,b]$ the sole root

$$\rho_c = \frac{b}{a}\,\frac{b(\sigma^c - p_e) - \sqrt{b^2(\sigma^c - p_e)^2 - a^2(\sigma^c - p_i)^2}}{\sigma^c - p_i}. \qquad (46)$$

It can be seen that when the ratio $(\sigma^c - p_e)/(\sigma^c - p_i)$ decreases from $(a^2+b^2)/2b^2$ to a/b, ρ_c correspondingly varies from a to b. Finally, the stress $\mathbf{T}$ having principal components:

$$\sigma_\rho(\rho) = \begin{cases} \dfrac{a}{\rho}(\sigma^c - p_i) - \sigma^c, & \rho \in [a, \rho_c], \\ \dfrac{a}{2}(\sigma^c - p_i)\left(\dfrac{\rho_c}{\rho^2} + \dfrac{1}{\rho_c}\right) - \sigma^c, & \rho \in [\rho_c, b]; \end{cases} \tag{47}$$

$$\sigma_\theta(\rho) = \begin{cases} -\sigma^c, & \rho \in [a, \rho_c], \\ \dfrac{a}{2}(\sigma^c - p_i)\left(-\dfrac{\rho_c}{\rho^2} + \dfrac{1}{\rho_c}\right) - \sigma^c, & \rho \in [\rho_c, b], \end{cases} \tag{48}$$

is statically admissible.

In agreement with the constitutive equations (1)–(5) and (7)–(9), in Ω_2 the fracture strain and the crushing strain are nil; the total deformation has components:

$$\varepsilon_\rho(\rho) = \frac{1+v}{2\mathrm{E}}\left\{(1-2v)\left[(\sigma^c - p_i)\frac{a}{\rho_c} - 2\sigma^c\right] + (\sigma^c - p_i)\frac{a\rho_c}{\rho^2}\right\}, \quad \rho \in [\rho_c, b], \tag{49}$$

$$\varepsilon_\theta(\rho) = \frac{1+v}{2\mathrm{E}}\left\{(1-2v)\left[(\sigma^c - p_i)\frac{a}{\rho_c} - 2\sigma^c\right] - (\sigma^c - p_i)\frac{a\rho_c}{\rho^2}\right\}, \quad \rho \in [\rho_c, b]; \tag{50}$$

the radial displacement is:

$$u(\rho) = \frac{1+v}{2\mathrm{E}}\left\{(1-2v)\left[(\sigma^c - p_i)\frac{a}{\rho_c} - 2\sigma^c\right]\rho - (\sigma^c - p_i)\frac{a\rho_c}{\rho}\right\}, \quad \rho \in [\rho_c, b].$$

In Ω_1 the fracture strain is nil and the total deformation has components:

$$\varepsilon_\rho(\rho) = \varepsilon_\rho^e(\rho) = \frac{1+v}{\mathrm{E}}\left[(1-v)(\sigma^c - p_i)\frac{a}{\rho} - (1-2v)\sigma^c\right], \quad \rho \in [a, \rho_c], \tag{51}$$

$$\varepsilon_\theta(\rho) = \varepsilon_\theta^e(\rho) + \varepsilon_\theta^c(\rho) = \frac{1+v}{\mathrm{E}}\left[-v(\sigma^c - p_i)\frac{a}{\rho} - (1-2v)\sigma^c\right] + \varepsilon_\theta^c(\rho), \quad \rho \in [a, \rho_c], \tag{52}$$

where the circumferential crushing strain ε_θ^c is a non-positive function of ρ which needs to be determined. The radial displacement, obtained by integrating ε_ρ is:

$$u(\rho) = \frac{1+v}{\mathrm{E}}[(1-v)(\sigma^c - p_i)a\ln\rho - (1-2v)\sigma^c\rho] + k, \quad \rho \in [a, \rho_c],$$

where $k = -[(1+v)/\mathrm{E}](\sigma^c - p_i)a[v + (1-v)\ln\rho_c]$ is a constant whose value is determined by imposing the continuity of the radial displacement at $\rho = \rho_c$. By virtue of eqn (52) we have:

$$\varepsilon_\theta^c(\rho) = \frac{1-v^2}{\mathrm{E}}\frac{a}{\rho}(\sigma^c - p_i)\ln\left(\frac{\rho}{\rho_c}\right), \quad \rho \in [a, \rho_c], \tag{53}$$

therefore the crushing strain is negative in Ω_1 and zero when $\rho = \rho_c$. It is interesting to remark that if $(a/b) < (\sigma^c - p_e)/(\sigma^c - p_i) \leqslant (a^2 + b^2)/2b^2$, besides the stress field, the strain and displacement fields are also unique, whereas if $(\sigma^c - p_e)/(\sigma^c - p_i) = (a/b)$, the displacement and thus the circumferential crushing strain are not unique and depend upon the constant k. If the ratio $(\sigma^c - p_e)/(\sigma^c - p_i)$ is less than the critical value a/b, there are no statically admissible stress fields.

Now, let us suppose

$$\frac{a^2+b^2}{2b^2} \leqslant \frac{p_e}{p_i} \leqslant 1.$$

In this case we have $\sigma_\theta^{(e)}(b) \leqslant \sigma_\theta^{(e)}(a) \leqslant -p_i$, and moreover, by virtue of the inequalities

$$\frac{\sigma^c - p_e}{\sigma^c - p_i} \geqslant \frac{p_e}{p_i} \geqslant \frac{a^2+b^2}{2b^2} \geqslant \frac{2a^2}{a^2+b^2},$$

the condition $\sigma_\theta^{(e)}(b) \geqslant -\sigma^c$ is always satisfied, so $\mathbf{T}^{(e)}$ is a statically admissible stress field.

Finally, we need to consider the case:

$$\frac{p_e}{p_i} \leqslant \frac{a^2+b^2}{2b^2}.$$

If p_e and p_i also satisfy the inequality $(p_e/p_i) \geqslant (a/b)$, then the semi-definite negative stress field $\mathbf{T}$ calculated by Bennati and Padovani (1992), having principal components:

$$\sigma_\rho(\rho) = \begin{cases} -\dfrac{ap_i}{\rho}, & \rho \in [a, \rho_t], \\ -\dfrac{ap_i}{2}\left(\dfrac{\rho_t}{\rho^2} + \dfrac{1}{\rho_t}\right), & \rho \in [\rho_t, b]; \end{cases} \tag{54}$$

$$\sigma_\theta(\rho) = \begin{cases} 0, & \rho \in [a, \rho_t], \\ \dfrac{ap_i}{2}\left(\dfrac{\rho_t}{\rho^2} - \dfrac{1}{\rho_t}\right), & \rho \in [\rho_t, b]; \end{cases} \tag{55}$$

is statically admissible, since $\sigma_\theta(a) \geqslant \sigma_\theta(b) \geqslant -p_e$. The transition radius from the region in which $\mathbf{E}^t \neq \mathbf{0}$ to the one in which $\mathbf{E}^t = \mathbf{0}$ is:

$$\rho_t = \frac{b}{a}\,\frac{bp_e - \sqrt{b^2p_e^2 - a^2p_i^2}}{p_i}, \tag{56}$$

in particular, if $(p_e/p_i) = (a/b)$, $\rho_t = b$ and if $(p_e/p_i) = (a^2+b^2)/2b^2$, then $\rho_t = a$. We can remark that, according to the elasticity of the material, the radius ρ_t depends only on the current values of the loads p_i and p_e. Therefore, contrary to what happens for elastic–plastic materials, one has no interest in considering cyclic loading processes. The crushing and radial fracture strains are both nil and the circumferential fracture strain is:

$$\varepsilon_\theta^t(\rho) = \begin{cases} \dfrac{1-\nu^2}{\mathrm{E}}\,\dfrac{ap_i}{\rho}\ln\left(\dfrac{\rho_t}{\rho}\right), & \rho \in [a, \rho_t], \\ 0, & \rho \in [\rho_t, b]. \end{cases} \tag{57}$$

Once p_i has been fixed, the cracked regions that is those where ε_θ^t is non-nil, can be increased or diminished, by decreasing or increasing p_e, respectively. Finally, for values of p_e/p_i less than a/b no statically admissible stress field exists. Now we increase the external pressure p_e from $(a/b)p_i$ to $(a/b)p_i + [(b-a)/b]\sigma^c$, while maintaining the internal pressure p_i

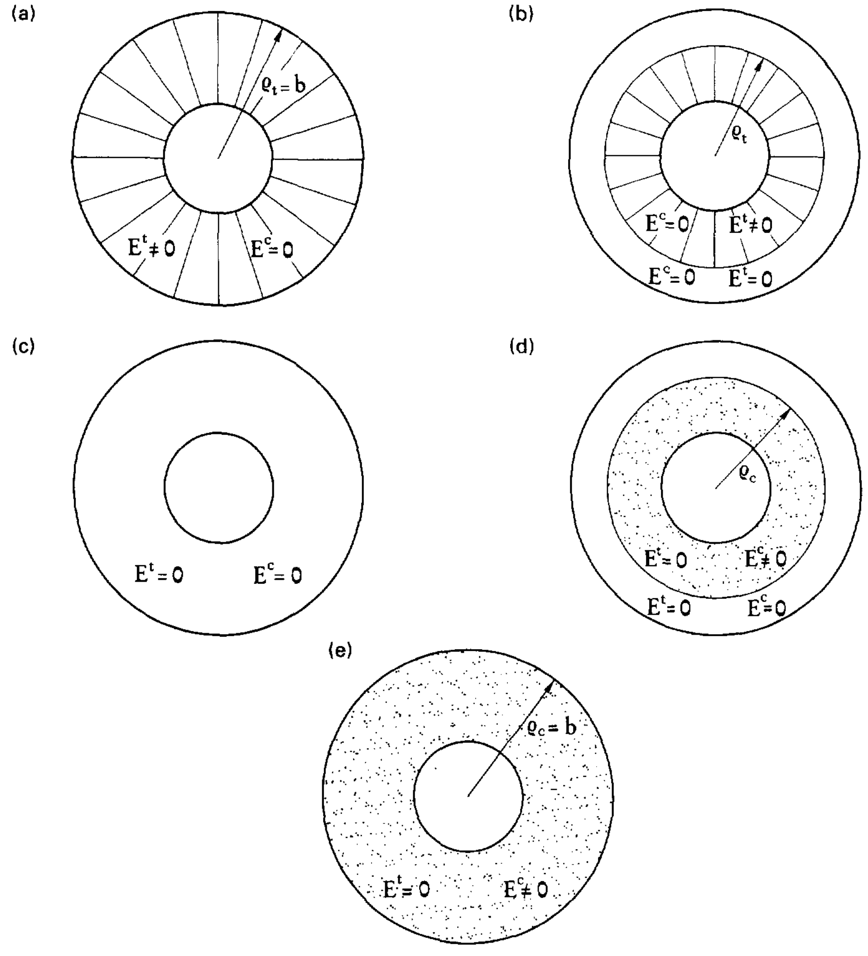

Fig. 3. Evolution of the inelastic strain for different values of p_e.

constant. Figure 3 shows the evolution of the inelastic strain for different values of p_e. When $p_e = (a/b)p_i$ (Fig. 3a), the crushing strain is nil and the fracture strain is non-zero throughout the circular ring; for $p_e \in [(a/b)p_i, ((a^2+b^2)/2b^2)p_i]$ (Fig. 3b), the crushing strain is still nil and the region in which the fracture strain is non-zero diminishes progressively and disappears when p_e falls within the interval $[((a^2+b^2)/2b^2)p_i, ((a^2+b^2)/2b^2)p_i + ((b^2-a^2)/2b^2)\sigma^c]$. In fact, for these values of p_e (Fig. 3c) the crushing and the fracture strain are zero. For p_e increasing from $((a^2+b^2)/2b^2)p_i + ((b^2-a^2)/2b^2)\sigma^c$ to $(a/b)p_i + ((b-a)/b)\sigma^c$ (Fig. 3d), the fracture strain remains equal to zero and the region in which the crushing strain is non-zero progressively extends and covers the whole of the circular ring when p_e reaches the value $(a/b)p_i + ((b-a)/b)\sigma^c$ (Fig. 3e). For values of p_e less than $(a/b)p_i$ and greater than $(a/b)p_i + ((b-a)/b)\sigma^c$ there are no statically admissible stress fields.

4.2. *The spherical container*

Let us consider a spherical container Ω_s made up of a BCS material with inner radius a and outer radius b, subjected to two uniform radial pressures: a pressure p_e acting on the external boundary and a pressure p_i acting on the internal boundary. Let $\{O, \rho, \theta, \varphi\}$ be a spherical reference system, with origin O coinciding with the centre of the container.

Bennati and Padovani (1992) have shown that if $(p_e/p_i) \geq (2a^3+b^3)/3b^3$, the stress field $\mathbf{T}^{(e)}$ corresponding to a linear elastic material and having the principal components:

$$\sigma_\rho^{(e)} = \frac{a^3b^3(p_e-p_i)}{b^3-a^3}\frac{1}{\rho^3}+\frac{p_ia^3-p_eb^3}{b^3-a^3},$$

$$\sigma_\theta^{(e)} = \sigma_\varphi^{(e)} = -\frac{a^3b^3(p_e-p_i)}{2(b^3-a^3)}\frac{1}{\rho^3}+\frac{p_ia^3-p_eb^3}{b^3-a^3}, \tag{58}$$

is negative semi-definite. First of all, let us suppose:

$$\frac{p_e}{p_i} \geqslant 1,$$

so the circumferential stress satisfies the boundary inequalities $\sigma_\theta^{(e)}(a) \leqslant \sigma_\theta^{(e)}(b) \leqslant -p_e$. Furthermore, if:

$$\frac{\sigma^c-p_e}{\sigma^c-p_i} \geqslant \frac{2a^3+b^3}{3b^3},$$

then the elastic solution (58) satisfies condition (8) and $\mathbf{T}^{(e)}$ is statically admissible. On the contrary, if:

$$\frac{\sigma^c-p_e}{\sigma^c-p_i} < \frac{2a^3+b^3}{3b^3},$$

then $\sigma_\theta^{(e)}(a) < -\sigma^c$ and $\mathbf{T}^{(e)}$ is not statically admissible. Using a procedure similar to that used for the circular ring, we may suppose that the spherical region $\Omega_{s1} = \{\rho;\rho\in[a,\rho_c]\}$, where ρ_c has to be determined, is subjected to the equilibrated stress field:

$$\sigma_\rho(\rho) = \frac{a^2}{\rho^2}(\sigma^c-p_i)-\sigma^c,$$

$$\sigma_\theta(\rho) = \sigma_\varphi(\rho) = -\sigma^c.$$

Consequently, the remaining spherical region $\Omega_{s2} = \{\rho;\rho\in[\rho_c,b]\}$ is subjected to the external pressure p_e and to the internal pressure $p_c = \sigma^c-(a^2/\rho_c^2)(\sigma^c-p_i)$. On the other hand, for continuity reasons, equalities $\sigma_\theta(\rho_c^+) = \sigma_\varphi(\rho_c^+) = -\sigma^c$ must hold. Finally, by virtue of eqn 53, we determine that if the ratio $(\sigma^c-p_e)/(\sigma^c-p_i)$ satisfies the inequalities:

$$\frac{a^2}{b^2} \leqslant \frac{\sigma^c-p_e}{\sigma^c-p_i} \leqslant \frac{2a^3+b^3}{3b^3},$$

a statically admissible stress field will have components:

$$\sigma_\rho(\rho) = \begin{cases} \dfrac{a^2}{\rho^2}(\sigma^c-p_i)-\sigma^c, & \rho\in[a,\rho_c], \\ \dfrac{a^2}{3}(\sigma^c-p_i)\left(\dfrac{2\rho_c}{\rho^3}+\dfrac{1}{\rho_c^2}\right)-\sigma^c, & \rho\in[\rho_c,b]; \end{cases} \tag{59}$$

$$\sigma_\theta(\rho) = \sigma_\varphi(\rho) = \begin{cases} -\sigma^c, & \rho\in[a,\rho_c], \\ \dfrac{a^2}{3}(\sigma^c-p_i)\left(-\dfrac{\rho_c}{\rho^3}+\dfrac{1}{\rho_c^2}\right)-\sigma^c, & \rho\in[\rho_c,b]. \end{cases} \tag{60}$$

The radius ρ_c, which separates the zone where $\mathbf{E}^c \neq \mathbf{0}$ from the zone in which $\mathbf{E}^c = \mathbf{0}$, is the sole real root belonging to $[a, b]$ of the third degree polynomial:

$$q(\rho) = 2a^2(\sigma^c - p_i)\rho^3 - 3b^3(\sigma^c - p_e)\rho^2 + a^2b^3(\sigma^c - p_i);$$

in particular if $(\sigma^c - p_e)/(\sigma^c - p_i) = (a^2/b^2)$, then $\rho_c = b$ and if $(\sigma^c - p_e)/(\sigma^c - p_i) = (2a^3 + b^3)/3b^3$, then $\rho_c = a$. In fact, it easy to verify that $q(a) \geqslant 0$ and $q(b) \leqslant 0$, moreover in the interval $[a, b]$, $q(\rho)$ is a decreasing function, thus there exist a unique $\rho_c \in [a, b]$ such that $q(\rho_c) = 0$. The value of ρ_c may be determined by using the well-known Cardano's formula, the calculation is here omitted for the sake of simplicity.

The circumferential displacement is nil for symmetry reasons, the radial displacement and the circumferential component of the crushing strain are univocally determined and have the expressions:

$$u(\rho) = \begin{cases} \dfrac{1}{\mathrm{E}}\left\{-(1-2\nu)\sigma^c\rho + (\sigma^c - p_i)a^2\left[(1-\nu)\dfrac{1}{\rho_c} - \dfrac{1}{\rho}\right]\right\}, & \rho \in [a, \rho_c], \\ \dfrac{1}{\mathrm{E}}\left\{(1-2\nu)\left[-\sigma^c + (\sigma^c - p_i)\dfrac{a^2}{3\rho_c^2}\right]\rho - (1+\nu)\dfrac{\rho_c a^2}{3\rho^2}(\sigma^c - p_i)\right\}, & \rho \in [\rho_c, b]; \end{cases} \tag{61}$$

$$\varepsilon_\theta^c(\rho) = \begin{cases} \dfrac{1-\nu}{\mathrm{E}}\dfrac{a^2}{\rho}(\sigma^c - p_i)\left(\dfrac{1}{\rho_c} - \dfrac{1}{\rho}\right), & \rho \in [a, \rho_c], \\ 0, & \rho \in [\rho_c, b]. \end{cases} \tag{62}$$

The fracture strain and the radial component of crushing strain are nil.

For values of $(\sigma^c - p_e)/(\sigma^c - p_i)$ less than a^2/b^2 there are no statically admissible stress fields. If p_e/p_i satisfies the inequalities:

$$\frac{2a^3 + b^3}{3b^3} \leqslant \frac{p_e}{p_i} \leqslant 1,$$

then $-p_e \geqslant \sigma_\theta^{(e)}(a) \geqslant \sigma_\theta^{(e)}(b)$ and $\mathbf{T}^{(e)}$ is a statically admissible stress.

Finally, the case in which

$$\frac{a^2}{b^2} \leqslant \frac{p_e}{p_i} \leqslant \frac{2a^3 + b^3}{3b^3}$$

has been already studied by Bennati and Padovani (1992); a statically admissible stress field has the principal components

$$\sigma_\rho(\rho) = \begin{cases} -\dfrac{a^2 p_i}{\rho^2}, & \rho \in [a, \rho_t], \\ -\dfrac{a^2 p_i}{3}\left(\dfrac{2\rho_t}{\rho^3} + \dfrac{1}{\rho_t^2}\right), & \rho \in [\rho_t, b]; \end{cases} \tag{63}$$

$$\sigma_\theta(\rho) = \begin{cases} 0, & \rho \in [a, \rho_t], \\ \dfrac{a^2 p_i}{3}\left(\dfrac{\rho_t}{\rho^3} - \dfrac{1}{\rho_t^2}\right), & \rho \in [\rho_t, b]; \end{cases} \tag{64}$$

where ρ_t is the sole real root belonging to $[a, b]$ of the cubic equation:

$$2a^2p_i\rho^3 - 3b^3p_e\rho^2 + a^2b^3p_i = 0$$

and separates the region in which the circumferential fracture strain is non-zero from the region in which it is zero. When p_e/p_i varies from $(2a^3+b^3)/3b^3$ to a^2/b^2, radius ρ_t correspondingly varies from a to b. The crushing strain is nil, the radial displacement is:

$$u(\rho) = \begin{cases} \dfrac{a^2 p_i}{E}\left[\dfrac{1}{\rho} - (1-v)\dfrac{1}{\rho_t}\right], & \rho \in [a, \rho_t], \\ \dfrac{p_i}{3E}\dfrac{a^2}{\rho_t^2}\left[(2v-1)\rho + (1+v)\dfrac{\rho_t^3}{\rho^2}\right], & \rho \in [\rho_t, b], \end{cases} \tag{65}$$

and the circumferential fracture strain is:

$$\varepsilon_\theta^t(\rho) = \begin{cases} \dfrac{1-v}{E}\dfrac{a^2}{\rho}p_i\left(\dfrac{1}{\rho} - \dfrac{1}{\rho_t}\right), & \rho \in [a, \rho_t], \\ 0, & \rho \in [\rho_t, b]. \end{cases} \tag{66}$$

For values of p_e/p_i less than a^2/b^2, there are no statically admissible stress fields.

5. THE NUMERICAL METHOD

In this section we calculate the derivative $D_E\hat{\mathbf{T}}$ of $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E})$ with respect to $\mathbf{E}$. Knowing this derivative allows calculation of the tangent matrix and determination of the displacements by solving a non-linear system obtained by discretisation into finite elements via the Newton–Raphson method.

The algorithm used for the numerical solution of the equilibrium problem in the presence of incremental loads has already been described by Lucchesi *et al.* (1994b) and is thus omitted here.

Differentiating $\hat{\mathbf{T}}$ with respect to $\mathbf{E}$ requires some preliminary results.

Let Sym* stand for the subset of Sym of all symmetric tensors having distinct eigenvalues. Given $\mathbf{A} \in \mathrm{Sym}^*$, let $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_1 < \lambda_2 < \lambda_3$ and $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ be the eigenvalues and the eigenvectors of $\mathbf{A}$, respectively.

Putting, for convenience,

$$\mathbf{G}_1 = \frac{1}{\sqrt{2}}(\mathbf{g}_1 \otimes \mathbf{g}_2 + \mathbf{g}_2 \otimes \mathbf{g}_1), \quad \mathbf{G}_2 = \frac{1}{\sqrt{2}}(\mathbf{g}_1 \otimes \mathbf{g}_3 + \mathbf{g}_3 \otimes \mathbf{g}_1),$$

$$\mathbf{G}_3 = \frac{1}{\sqrt{2}}(\mathbf{g}_2 \otimes \mathbf{g}_3 + \mathbf{g}_3 \otimes \mathbf{g}_2),$$

we propose to prove the following:†

$$D_{\mathbf{A}}\lambda_1 = \mathbf{g}_1 \otimes \mathbf{g}_1, \tag{67}$$

$$D_{\mathbf{A}}\lambda_2 = \mathbf{g}_2 \otimes \mathbf{g}_2, \tag{68}$$

$$D_{\mathbf{A}}\lambda_3 = \mathbf{g}_3 \otimes \mathbf{g}_3; \tag{69}$$

† Here $D_{\mathbf{A}}\lambda_i$ is the derivative with respect to $\mathbf{A}$ of the function $\lambda_i: \mathrm{Sym}^* \to \mathbb{R}, \mathbf{A} | \to \lambda_i(\mathbf{A})$; analogously $D_{\mathbf{A}}\mathbf{g}_i \otimes \mathbf{g}_i$ is the derivative with respect to $\mathbf{A}$ of the function $\mathbf{g}_i \otimes \mathbf{g}_i$. This last function is well defined since, by virtue of the fact that the eigenvalues of $\mathbf{A}$ are distinct, the eigenvectors $\mathbf{g}_i$ are uniquely determined from the relations $\mathbf{A}\mathbf{g}_i = \lambda_i\mathbf{g}_i$, $i = 1, 2, 3$.

$$D_{\mathbf{A}}\mathbf{g}_1 \otimes \mathbf{g}_1 = \frac{1}{\lambda_1-\lambda_2}\mathbf{G}_1 \otimes \mathbf{G}_1 + \frac{1}{\lambda_1-\lambda_3}\mathbf{G}_2 \otimes \mathbf{G}_2, \tag{70}$$

$$D_{\mathbf{A}}\mathbf{g}_2 \otimes \mathbf{g}_2 = \frac{1}{\lambda_2-\lambda_1}\mathbf{G}_1 \otimes \mathbf{G}_1 + \frac{1}{\lambda_2-\lambda_3}\mathbf{G}_3 \otimes \mathbf{G}_3, \tag{71}$$

$$D_{\mathbf{A}}\mathbf{g}_3 \otimes \mathbf{g}_3 = \frac{1}{\lambda_3-\lambda_1}\mathbf{G}_2 \otimes \mathbf{G}_2 + \frac{1}{\lambda_3-\lambda_2}\mathbf{G}_3 \otimes \mathbf{G}_3. \tag{72}$$

It is sufficient to prove eqns (67) and (70), because the other relations can be proven in a similar way. Let us consider $\mathbf{A} \in \mathrm{Sym}^*$, $\mathbf{H} \in \mathrm{Sym}$ to be fixed and $\alpha \in \mathbb{R}$; let $\lambda_1(\alpha)$ and $\mathbf{g}_1(\alpha)$ be the smallest eigenvalue and the corresponding eigenvector of $\mathbf{A}+\alpha\mathbf{H}$, respectively:

$$(\mathbf{A}+\alpha\mathbf{H})\mathbf{g}_1(\alpha) = \lambda_1(\alpha)\mathbf{g}_1(\alpha). \tag{73}$$

Since we are interested in the behaviour of $\lambda_1(\alpha)$ and $\mathbf{g}_1(\alpha)$ for α near zero, within an error of order $o(\alpha)$ we can put:

$$\lambda_1(\alpha) = \lambda_1 + \dot{\lambda}_1(0), \quad \text{and} \quad \mathbf{g}_1(\alpha) = \mathbf{g}_1 + \dot{\mathbf{g}}_1(0), \tag{74}$$

where $\lambda_1 = \lambda_1(0)$, $\mathbf{g}_1 = \mathbf{g}_1(0)$ and the superimposed dot $\cdot$ denotes differentiation with respect to α. By substituting eqn (74) in (73) we obtain:

$$\mathbf{A}\dot{\mathbf{g}}_1(0) + \mathbf{H}\mathbf{g}_1 = \dot{\lambda}_1(0)\mathbf{g}_1 + \lambda_1\dot{\mathbf{g}}_1(0). \tag{75}$$

Since $\mathbf{g}_1 \cdot \mathbf{g}_1 = 1$, then $\dot{\mathbf{g}}_1(0) \cdot \mathbf{g}_1 = 0$; thus if we multiply eqn (75) by $\mathbf{g}_1$ we have

$$\dot{\lambda}_1(0) = \mathbf{g}_1 \cdot \mathbf{H}\mathbf{g}_1 = \mathbf{g}_1 \otimes \mathbf{g}_1 \cdot \mathbf{H}. \tag{76}$$

Because, for every $\mathbf{H}$ in Sym we can write

$$\dot{\lambda}_1(0) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\lambda_1(\mathbf{A}+\alpha\mathbf{H})|_{\alpha=0} = D_{\mathbf{A}}\lambda_1 \cdot \mathbf{H},$$

by virtue of eqn (76) we obtain eqn (67).

In order to calculate the derivative of $\mathbf{g}_1 \otimes \mathbf{g}_1$, we have to calculate the derivative of $\mathbf{g}_1$. To this end, by substituting eqn (76) into eqn (75), we obtain:

$$\mathbf{A}\dot{\mathbf{g}}_1(0) + \mathbf{H}\mathbf{g}_1 = (\mathbf{g}_1 \otimes \mathbf{g}_1 \cdot \mathbf{H})\mathbf{g}_1 + \lambda_1\dot{\mathbf{g}}_1(0). \tag{77}$$

Since $\mathbf{g}_1$ and $\dot{\mathbf{g}}_1(0)$ are orthogonal, we can write:

$$\dot{\mathbf{g}}_1(0) = \chi\mathbf{g}_2 + \xi\mathbf{g}_3, \tag{78}$$

where χ and ξ are scalars which depend on $\mathbf{A}$. By substituting eqn (78) into eqn (77) the relation

$$\chi(\lambda_2-\lambda_1)\mathbf{g}_2 + \xi(\lambda_3-\lambda_1)\mathbf{g}_3 = (\mathbf{g}_1 \otimes \mathbf{g}_1 \cdot \mathbf{H})\mathbf{g}_1 - \mathbf{H}\mathbf{g}_1 \tag{79}$$

follows. Multiplying eqn (79) by $\mathbf{g}_2$ and by $\mathbf{g}_3$, we obtain respectively,

$$\chi = \frac{1}{\lambda_1 - \lambda_2} \mathbf{g}_1 \otimes \mathbf{g}_2 \cdot \mathbf{H},$$

$$\xi = \frac{1}{\lambda_1 - \lambda_3} \mathbf{g}_1 \otimes \mathbf{g}_3 \cdot \mathbf{H}. \tag{80}$$

Thus, from eqn (78) and eqn (80), by virtue of the symmetry of **H**, we have:†

$$\dot{\mathbf{g}}_1(0) = \frac{\mathrm{d}}{\mathrm{d}\alpha} \mathbf{g}_1(\mathbf{A}+\alpha\mathbf{H})|_{\alpha=0} = D_{\mathbf{A}}\mathbf{g}_1[\mathbf{H}] = \frac{1}{2(\lambda_1-\lambda_2)}(\mathbf{g}_2 \otimes \mathbf{g}_1 \otimes \mathbf{g}_2 + \mathbf{g}_2 \otimes \mathbf{g}_2 \otimes \mathbf{g}_1)[\mathbf{H}]$$
$$+ \frac{1}{2(\lambda_1-\lambda_3)}(\mathbf{g}_3 \otimes \mathbf{g}_1 \otimes \mathbf{g}_3 + \mathbf{g}_3 \otimes \mathbf{g}_3 \otimes \mathbf{g}_1)[\mathbf{H}]. \tag{81}$$

The desired result follows from the relation:

$$D_{\mathbf{A}}\mathbf{g}_1 \otimes \mathbf{g}_1[\mathbf{H}] = D_{\mathbf{A}}\mathbf{g}_1[\mathbf{H}] \otimes \mathbf{g}_1 + \mathbf{g}_1 \otimes D_{\mathbf{A}}\mathbf{g}_1[\mathbf{H}].$$

Now we are in a position to calculate the derivative of the stress with respect to the total deformation in the 10 regions $\mathscr{R}_i$. Let us consider the orthonormal basis of Sym:

$$\begin{aligned}
\mathbf{O}_1 &= \mathbf{q}_1 \otimes \mathbf{q}_1, \\
\mathbf{O}_2 &= \mathbf{q}_2 \otimes \mathbf{q}_2, \\
\mathbf{O}_3 &= \mathbf{q}_3 \otimes \mathbf{q}_3, \\
\mathbf{O}_4 &= \frac{1}{\sqrt{2}}(\mathbf{q}_1 \otimes \mathbf{q}_2 + \mathbf{q}_2 \otimes \mathbf{q}_1), \\
\mathbf{O}_5 &= \frac{1}{\sqrt{2}}(\mathbf{q}_1 \otimes \mathbf{q}_3 + \mathbf{q}_3 \otimes \mathbf{q}_1), \\
\mathbf{O}_6 &= \frac{1}{\sqrt{2}}(\mathbf{q}_2 \otimes \mathbf{q}_3 + \mathbf{q}_3 \otimes \mathbf{q}_2),
\end{aligned} \tag{82}$$

and the spectral representation of **T**:

$$\mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{O}_i \tag{83}$$

where t_1, t_2 and t_3 are given in eqns (11)–(20). From eqns (11), (14) and (17) it follows that the expression of $D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E})$ for **E** belonging to $\mathscr{R}_1$, $\mathscr{R}_4$ and $\mathscr{R}_7$ can be easily calculated; the calculation of $D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E})$ when **E** belongs to the seven other regions is slightly more complex and requires differentiating eqn (83). In order to differentiate eqn (83) with respect to **E** we must use the previously calculated derivatives of the eigenvalues of **E** and the tensors $\mathbf{O}_1$, $\mathbf{O}_2$ and $\mathbf{O}_3$ with respect to **E**.

As a single example, we shall calculate $D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E})$ when $\mathbf{E} \in \mathscr{R}_2$, where $e_1 < e_2 \leqslant e_3$. Let us begin by supposing $e_1 < e_2 < e_3$; from eqns (83), (22) and (67)–(72), using the relation:

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = D_{\mathbf{E}}t_1 \otimes \mathbf{O}_1 + t_1 D_{\mathbf{E}}\mathbf{O}_1 + D_{\mathbf{E}}t_2 \otimes \mathbf{O}_2 + t_2 D_{\mathbf{E}}\mathbf{O}_2 + D_{\mathbf{E}}t_3 \otimes \mathbf{O}_3 + t_3 D_{\mathbf{E}}\mathbf{O}_3,$$

we obtain:

† Here, given $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathscr{V}$, $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$ denotes the third-order tensor defined by $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}\ \mathbf{H} = (\mathbf{v} \otimes \mathbf{w} \cdot \mathbf{H})\ \mathbf{u}$, $\mathbf{H} \in \mathrm{Lin}$.

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{2\mu}{2+\alpha}\frac{\varepsilon^{c}+2(1+\alpha)e_2+\alpha e_3}{e_2-e_1}\mathbf{O}_4\otimes\mathbf{O}_4 + \frac{2\mu}{2+\alpha}\frac{\varepsilon^{c}+2(1+\alpha)e_3+\alpha e_2}{e_3-e_1}\mathbf{O}_5\otimes\mathbf{O}_5 + f(e_2,e_3)\mathbf{O}_6\otimes\mathbf{O}_6 + \frac{\mu(2+3\alpha)}{2+\alpha}(\mathbf{O}_2+\mathbf{O}_3)\otimes(\mathbf{O}_2+\mathbf{O}_3)+\mu(\mathbf{O}_2-\mathbf{O}_3)\otimes(\mathbf{O}_2-\mathbf{O}_3), \quad (84)$$

where $f(e_2, e_3) = 2\mu(e_3-e_2)/(e_3-e_2)$. When e_3-e_2 goes to zero, $f(e_2, e_3)$ converges on 2μ and then eqn (84), with $f(e_2, e_3) = 2\mu$, holds also when $e_3 = e_2$.

Finally, we summarise the expression of $D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E})$ in the 10 regions $\mathscr{R}_i$:

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = 2\mu\mathbb{1}+\lambda\mathbf{I}\otimes\mathbf{I}, \quad \mathbf{E}\in\mathscr{R}_1, \quad (85)$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{2\mu}{2+\alpha}\frac{\varepsilon^{c}+2(1+\alpha)e_2+\alpha e_3}{e_2-e_1}\mathbf{O}_4\otimes\mathbf{O}_4 + \frac{2\mu}{2+\alpha}\frac{\varepsilon^{c}+2(1+\alpha)e_3+\alpha e_2}{e_3-e_1}\mathbf{O}_5\otimes\mathbf{O}_5+2\mu\mathbf{O}_6\otimes\mathbf{O}_6 + \frac{\mu(2+3\alpha)}{2+\alpha}(\mathbf{O}_2+\mathbf{O}_3)\otimes(\mathbf{O}_2+\mathbf{O}_3)+\mu(\mathbf{O}_2-\mathbf{O}_3)\otimes(\mathbf{O}_2-\mathbf{O}_3), \quad \mathbf{E}\in\mathscr{R}_2, \quad (86)$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{1+\alpha}\frac{\varepsilon^{c}+(2+3\alpha)e_3}{e_3-e_1}\mathbf{O}_5\otimes\mathbf{O}_5 + \frac{\mu}{1+\alpha}\frac{\varepsilon^{c}+(2+3\alpha)e_3}{e_3-e_2}\mathbf{O}_6\otimes\mathbf{O}_6+E\mathbf{O}_3\otimes\mathbf{O}_3, \quad \mathbf{E}\in\mathscr{R}_3, \quad (87)$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}, \quad \mathbf{E}\in\mathscr{R}_4, \quad (88)$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = 2\mu\mathbf{O}_4\otimes\mathbf{O}_4+\frac{2\mu}{2+\alpha}\frac{\varepsilon^{t}-2(1+\alpha)e_1-\alpha e_2}{e_3-e_1}\mathbf{O}_5\otimes\mathbf{O}_5 + \frac{2\mu}{2+\alpha}\frac{\varepsilon^{t}-2(1+\alpha)e_2-\alpha e_1}{e_3-e_2}\mathbf{O}_6\otimes\mathbf{O}_6 + \frac{\mu(2+3\alpha)}{2+\alpha}(\mathbf{O}_1+\mathbf{O}_2)\otimes(\mathbf{O}_1+\mathbf{O}_2)+\mu(\mathbf{O}_1-\mathbf{O}_2)\otimes(\mathbf{O}_1-\mathbf{O}_2), \quad \mathbf{E}\in\mathscr{R}_5, \quad (89)$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{1+\alpha}\frac{\varepsilon^{t}-(2+3\alpha)e_1}{e_2-e_1}\mathbf{O}_4\otimes\mathbf{O}_4 + \frac{\mu}{1+\alpha}\frac{\varepsilon^{t}-(2+3\alpha)e_1}{e_3-e_1}\mathbf{O}_5\otimes\mathbf{O}_5+E\mathbf{O}_1\otimes\mathbf{O}_1, \quad \mathbf{E}\in\mathscr{R}_6, \quad (90)$$

$$\mathbf{D}_{E}\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}, \quad \mathbf{E}\in\mathscr{R}_7, \quad (91)$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{\sigma^{t}+\sigma^{c}}{e_2-e_1}\mathbf{O}_4\otimes\mathbf{O}_4+\frac{\sigma^{t}+\sigma^{c}}{e_3-e_1}\mathbf{O}_5\otimes\mathbf{O}_5, \quad \mathbf{E}\in\mathscr{R}_8, \quad (92)$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{\sigma^{t}+\sigma^{c}}{e_3-e_1}\mathbf{O}_5\otimes\mathbf{O}_5+\frac{\sigma^{t}+\sigma^{c}}{e_3-e_2}\mathbf{O}_6\otimes\mathbf{O}_6, \quad \mathbf{E}\in\mathscr{R}_9, \quad (93)$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{2(1+\alpha)}\frac{\alpha\varepsilon^{t}+(2+\alpha)\varepsilon^{c}+2(2+3\alpha)e_2}{e_2-e_1}\mathbf{O}_4\otimes\mathbf{O}_4 + \frac{\sigma^{t}+\sigma^{c}}{e_3-e_1}\mathbf{O}_5\otimes\mathbf{O}_5 + \frac{\mu}{2(1+\alpha)}\frac{\alpha\varepsilon^{c}+(2+\alpha)\varepsilon^{t}-2(2+3\alpha)e_2}{e_3-e_2}\mathbf{O}_6\otimes\mathbf{O}_6 + \frac{\mu(2+3\alpha)}{1+\alpha}\mathbf{O}_2\otimes\mathbf{O}_2, \quad \mathbf{E}\in\mathscr{R}_{10}, \tag{94}$$

where $\mathbb{1}$ and $\mathbb{O}$ are the fourth-order identity tensor and the fourth-order null tensor, respectively. It is note-worthy that the expressions given in eqns (86)–(94) are the spectral representations of $D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E})$ in the nine regions $\mathscr{R}_2$–$\mathscr{R}_{10}$. Moreover, it can be easily verified that the eigenvalues of $D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E})$ are non-negative and so the strain-energy density $\Psi(\mathbf{E}) = \frac{1}{2}\hat{\mathbf{T}}(\mathbf{E})\cdot\mathbf{E}$ is a convex function. The same result has been proven by Del Piero (1989) for materials not supporting tension and infinitely resistant to compression.

We conclude this section by listing the expression for the derivative of the stress for plane strain and plane stress.

For the plane case, let $e_1 < e_2$ be the eigenvalues of $\mathbf{E}$ and $\mathbf{q}_1$ and $\mathbf{q}_2$ be the corresponding eigenvectors, putting:

$$\mathbf{O}_1 = \mathbf{q}_1\otimes\mathbf{q}_1,$$
$$\mathbf{O}_2 = \mathbf{q}_2\otimes\mathbf{q}_2,$$
$$\mathbf{O}_3 = \frac{1}{\sqrt{2}}(\mathbf{q}_1\otimes\mathbf{q}_2+\mathbf{q}_2\otimes\mathbf{q}_1),$$

we have:

$$D_{\mathbf{E}}e_1 = \mathbf{O}_1, \qquad D_{\mathbf{E}}e_2 = \mathbf{O}_2,$$
$$D_{\mathbf{E}}\mathbf{O}_1 = \frac{1}{e_1-e_2}\mathbf{O}_3\otimes\mathbf{O}_3, \quad D_{\mathbf{E}}\mathbf{O}_2 = \frac{1}{e_2-e_1}\mathbf{O}_3\otimes\mathbf{O}_3.$$

For plane strain, the derivatives of $\mathbf{T}$ in the six regions $\mathscr{S}_1$–$\mathscr{S}_6$ are

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = 2\mu\mathbb{1}+\lambda\mathbf{I}\otimes\mathbf{I}, \quad \mathbf{E}\in\mathscr{S}_1, \tag{95}$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}, \quad \mathbf{E}\in\mathscr{S}_2, \tag{96}$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{2\mu}{2+\alpha}\frac{\varepsilon^{t}-2(1+\alpha)e_1}{e_2-e_1}\mathbf{O}_3\otimes\mathbf{O}_3+\frac{4\mu(1+\alpha)}{2+\alpha}\mathbf{O}_1\otimes\mathbf{O}_1, \quad \mathbf{E}\in\mathscr{S}_3, \tag{97}$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{2\mu}{2+\alpha}\frac{\varepsilon^{c}+2(1+\alpha)e_2}{e_2-e_1}\mathbf{O}_3\otimes\mathbf{O}_3+\frac{4\mu(1+\alpha)}{2+\alpha}\mathbf{O}_2\otimes\mathbf{O}_2, \quad \mathbf{E}\in\mathscr{S}_4, \tag{98}$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}, \quad \mathbf{E}\in\mathscr{S}_5, \tag{99}$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{\sigma^{c}+\sigma^{t}}{e_2-e_1}\mathbf{O}_3\otimes\mathbf{O}_3, \quad \mathbf{E}\in\mathscr{S}_6. \tag{100}$$

For plane stress we have:

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = 2\mu\mathbb{1} + \frac{2\mu\alpha}{2+\alpha}\mathbf{I}\otimes\mathbf{I}, \quad \mathbf{E}\in\mathscr{T}_1, \tag{101}$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}, \quad \mathbf{E}\in\mathscr{T}_2, \tag{102}$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{2(1+\alpha)}\frac{(2+\alpha)\varepsilon^{t}-2(2+3\alpha)e_1}{e_2-e_1}\mathbf{O}_3\otimes\mathbf{O}_3+\mathrm{E}\mathbf{O}_1\otimes\mathbf{O}_1, \quad \mathbf{E}\in\mathscr{T}_3, \tag{103}$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{\mu}{2(1+\alpha)}\frac{(2+\alpha)\varepsilon^{c}+2(2+3\alpha)e_2}{e_2-e_1}\mathbf{O}_3\otimes\mathbf{O}_3+\mathrm{E}\mathbf{O}_2\otimes\mathbf{O}_2, \quad \mathbf{E}\in\mathscr{T}_4, \tag{104}$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \mathbb{O}, \quad \mathbf{E}\in\mathscr{T}_5, \tag{105}$$

$$D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E}) = \frac{\sigma^{c}+\sigma^{t}}{e_2-e_1}\mathbf{O}_3\otimes\mathbf{O}_3, \quad \mathbf{E}\in\mathscr{T}_6. \tag{106}$$

6. NUMERICAL EXAMPLES

6.1. *The circular ring*

In this section we numerically solve the problem of the circular ring considered in Section 4. The finite element analysis is performed using the calculus scheme described by Lucchesi *et al.* (1994b), by means of the tangent stiffness matrix calculated with the help of the fourth-order tensor $D_{\mathbf{E}}\hat{\mathbf{T}}(\mathbf{E})$ deduced in the previous section. For the numerical calculation of the solution, the following values of the constants have been used:

$a = 1$ m,
$b = 1.5$ m,
$p_i = 0.1$ MPa,
$p_e = 0.23$ MPa,
$\sigma^c = 0.5$ MPa,
$\nu = 0.1$
$\mathrm{E} = 5000$ MPa.

In this case the ratio $(\sigma^c - p_e)/(\sigma^c - p_i) = 0.675$ lies within the interval $[(a/b),(a^2+b^2)/2b^2] = [0.667, 0.722]$ and the transition radius is approximately $\rho_c = 1.28$ m. For symmetry reasons, only a quarter of the circular ring was studied, and this was discretised into 400 eight-node elements; convergence was reached in three iterations. Figures 4, 5, and 6 show the behaviour of the radial stress, circumferential stress and

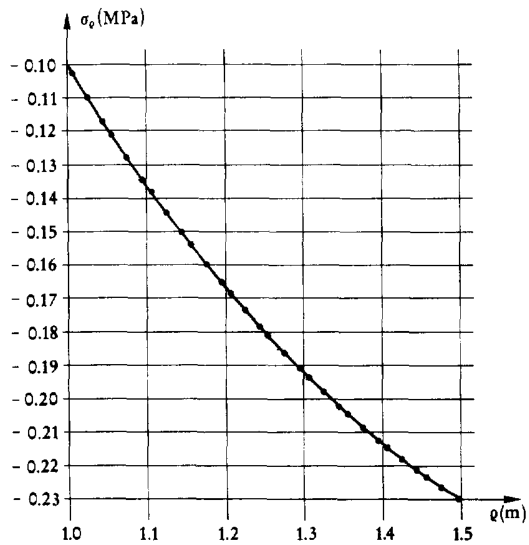

Fig. 4. Radial stress vs ρ.

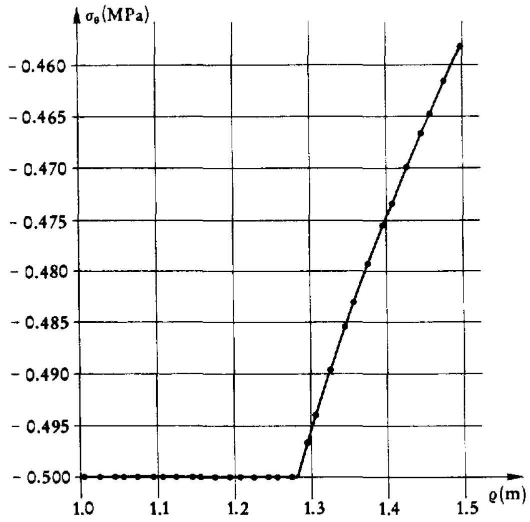

Fig. 5. Circumferential stress vs ρ.

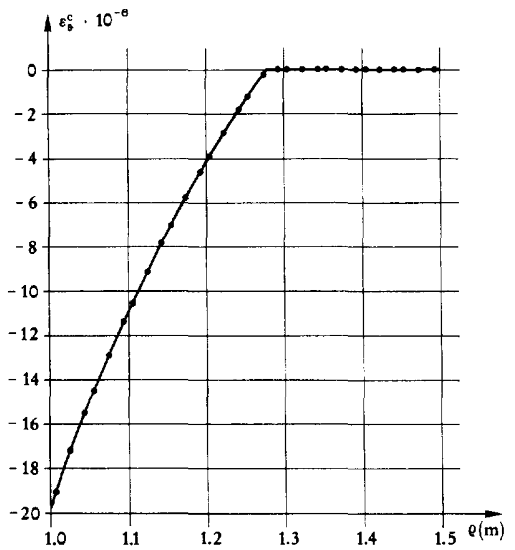

Fig. 6. Circumferential crushing strain vs ρ.

circumferential crushing strain. The continuous line represents the exact solution, the bold points, the numerical solution.

The circular ring was successively subjected to a load process with $p_i = 0.1$ MPa and p_e increasing from $p_{e0} = 0.0667$ MPa to $p_{ef} = 0.2333$ MPa. In Fig. 7 the behaviour of radius ρ^*, which separates the region in which the inelastic deformation $\mathbf{E}^t+\mathbf{E}^c$ is non-zero from the region in which $\mathbf{E}^t+\mathbf{E}^c = \mathbf{0}$, is shown. In accordance with eqns (56) and (46), the expression of ρ^* is:

$$\rho^*(p_e/p_i) = \begin{cases} 1.5\left(1.5\dfrac{p_e}{p_i} - \sqrt{2.25\dfrac{p_e^2}{p_i^2} - 1}\right), & p_e/p_i \in [0.667, 0.722], \\ 1, & p_e/p_i \in [0.722, 2.111], \\ 0.375\left[1.5\left(5-\dfrac{p_e}{p_i}\right) - \sqrt{2.25\left(5-\dfrac{p_e}{p_i}\right)^2 - 16}\right], & p_e/p_i \in [2.111, 2.333]. \end{cases}$$

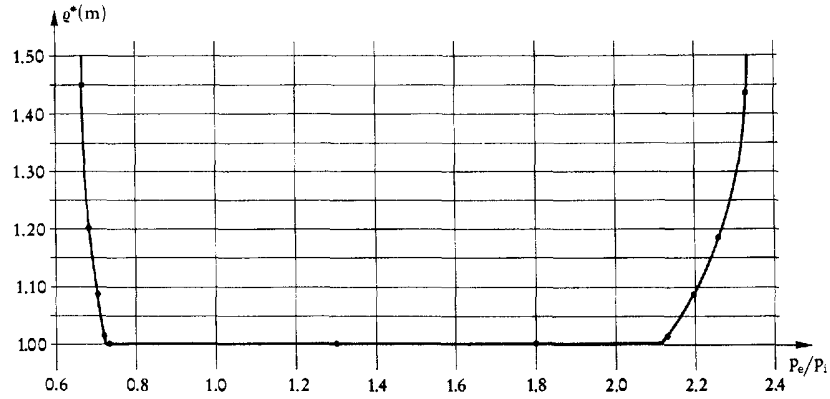

Fig. 7. Transition radius ρ^* vs p_e/p_i.

6.2. *Mosca's bridge*

The Mosca's bridge over the Doria Riparia, Turin, was constructed in 1827. It consists of 93 voussoirs made of Malanaggio granite, has a span of 45 m, an intrados rise of 5.5 m and thickness varying from 2 m at the springing to 1.5 m at the crown (Fig. 8). Mortar has been interposed only in the first 11 joints at the springings and the 22 joints nearest the crown. The bridge has been studied by Castigliano (1879) with the goal of verifying the advantageous effect of mortar joints on the behaviour of the line of thrust. In fact Castigliano has proven that when the mortar is accounted for, the line of thrust is contained entirely within the middle-third, while considering the arch ring as a monolith leads to a value of eccentricity e at the springing of 0.557 m, as opposed to a value of the middle-sixth of 0.33 m, corresponding to an opening of 0.682 m at the extrados. In order to obtain this last result, Castigliano considered a linear elastic material and used an iterative procedure.

Our goal is to determine the line of thrust by using the numerical method described in this paper under assumptions of infinite crushing stress σ^c and nil fracture stress σ^t. The arch ring is considered to be a monolith subjected to a plane strain; it is discretized into 800 elements with four nodes and four Gauss points, disposed in eight longitudinal lines, each consisting of 100 elements. The amount of loads is equal to that considered by Castigliano in his study, but rather than concentrating the total load in 12 points, we have assigned the weight of the arch ring as a body force and the rest of the load (the permanent load and the overload) has been distributed on the extrados. Figure 9 shows the behaviour of the line of thrust;† it is contained entirely within the middle-third except in the region delimited by the springing and the normal section 2.87 m from the springing, that is, the approximate region of the first six voussoirs. In particular, the eccentricity value at the springing is 0.616 m, with a corresponding opening of 0.848 m at the extrados. The horizontal component of thrust at the abutments is 3.963×10^6 N m^{-1}; this is 17% higher

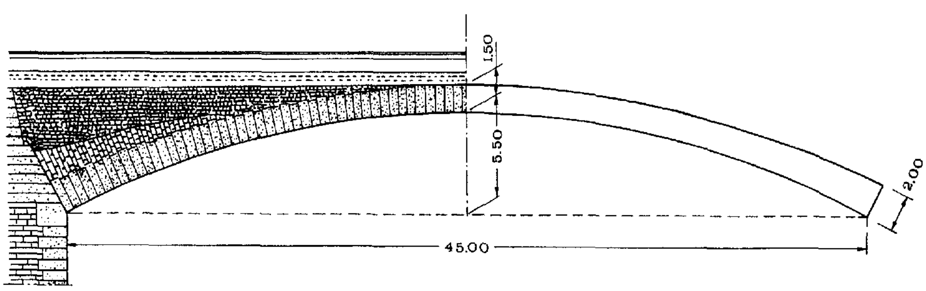

Fig. 8. Mosca's bridge.

† Stress state being determined by using the finite element method, we calculate the normal force N and the bending moment M in each normal section of the arch by means of an integration composite trapezoidal open rule using 50 intervals. In order to draw the line of thrust, the eccentricity e (i.e. the signed distance of the line of thrust from the mean line of the arch) is obtained from the well-known relation $e = M/N$.

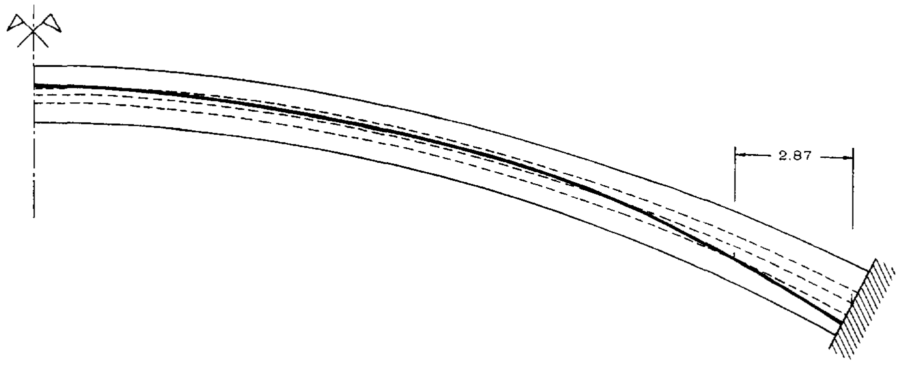

Fig. 9. The line of thrust for Mosca's bridge.

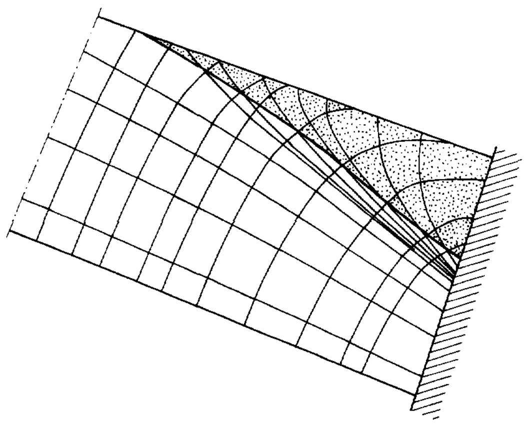

Fig. 10. The isostatic lines near the springing.

than Castigliano's result (3.2728×10^6 N m^{-1}). The greatest compression stress is obtained at the extrados in the springing and has the value of 7.6 MPa. The region characterised by the openings is illustrated in Fig. 10, where the isostatic lines are also drawn.

6.3. *The three-dimensional arch*

Let us consider the reduced circular arch whose springings are fixed, shown in Fig. 11. The arch is subjected to its own weight and a load p, constant per unit span, distributed along the extrados. For symmetry reasons, only a quarter of the structure was studied and this was discretized into 300 isoparametric three-dimensional elements with 20 nodes and 27 Gauss points. We suppose that the material constituting the arch is not resistant to traction ($\sigma^t = 0$) and has a crushing stress $\sigma^c = 8.82$ MPa. The distributed load is progressively increased until the value p_c, beyond which the convergence cannot be reached; p_c, interpreted here as collapse load, resulted equal to 0.405 MPa.

Collapse occurs because of the formation of a number of hinges sufficient to render the structure labile. The constitutive characteristics of the material suggest supposing that at the instant of collapse in the normal sections of the arch where there are the hinges, the normal stress σ is constant and equal to σ^c in an interval having an extremum coinciding with the intrados or the extrados and nil elsewhere. Figure 12 where $\sigma = -\sigma^c$ for $d \leqslant y \leqslant h/2$ and $\sigma = 0$ for $-h/2 \leqslant y \leqslant d$, shows one of these situations. The straight line parallel to the x axis and at a distance equal to d from it, is called the *neutral axis*.

Let us suppose that a section of the arch is at the limit state. Let N, M and $e = M/N$ be the normal force, bending moment and eccentricity, respectively; from Fig. 12 we

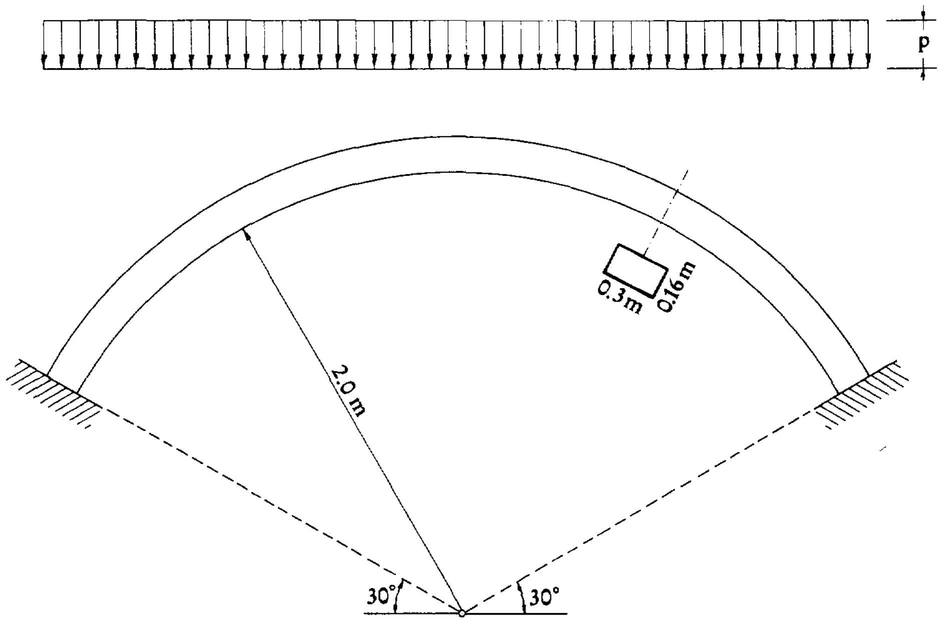

Fig. 11. The reduced circular arch.

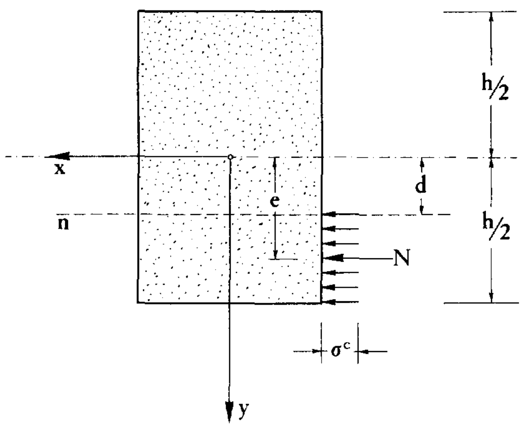

Fig. 12. Stress distribution in a normal section which is a hinge site.

deduce:

$$N = -2\sigma^c\left(\frac{h}{2} - |e|\right), \tag{107}$$

from which, putting $N^c = -h\sigma^c$, we obtain:

$$d = -\frac{h}{2}\left(1 - \frac{2N}{N^c}\right)\frac{|M|}{M}, \tag{108}$$

$$|M| = -\frac{Nh}{2}\left(1 - \frac{N}{N^c}\right) \tag{109}$$

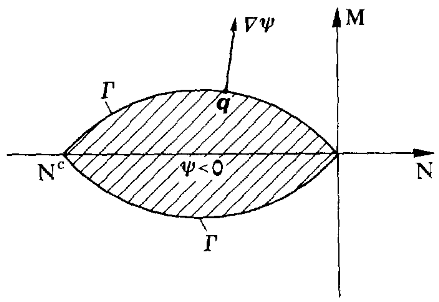

Fig. 13. The admissible region.

and

$$|e| = -\frac{|M|}{N} = \frac{h}{2}\left(1-\frac{N}{N^c}\right). \tag{110}$$

Then let us consider any section, not necessarily at the limit state; from eqn (110) we deduce

$$|e| \leqslant \frac{h}{2}\left(1-\frac{N}{N^c}\right), \tag{111}$$

therefore, that for each plane parallel to the mean plane, the line of thrust is contained entirely within the region of the arch delimited by the two curves having equations

$$y = -\frac{h}{2}\left(1-\frac{N}{N^c}\right), \quad y = \frac{h}{2}\left(1-\frac{N}{N^c}\right). \tag{112}$$

For each normal section, let us put:

$$\psi(N, M) = M + \frac{Nh}{2}\left(1-\frac{N}{N^c}\right)\frac{|M|}{M}. \tag{113}$$

The *admissible region*, defined by the relation $\psi(N, M) \leqslant 0$, is constituted by all pairs $\mathbf{q} = (N, M)$ which are compatible with the constitutive properties of the material; the curve Γ having equation $\psi(N, M) = 0$ is called *limit curve* (Fig. 13). In order for a section to be a hinge site, it is necessary that the relative normal force N and bending moment M belong to Γ; in this case the line of thrust is tangent to one of the two curves defined in eqn (112). Let us now suppose that $\mathbf{q} = (N, M)$ belongs to Γ and let us put $\alpha = (\delta, \varphi)$, where δ is the extension of the mean line of the arch and φ is the relative rotation of the section (Fig. 14). It is easily verified that from relation (9) it follows that the generalised displacement α is

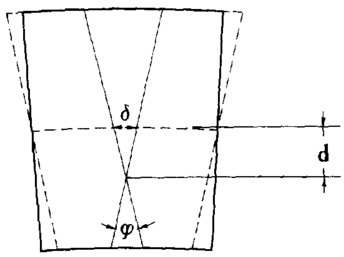

Fig. 14. Generalised displacement of a normal section.

kinematically admissible if and only if it has the same direction of the gradient of ψ calculated at $\mathbf{q}$. Thus, since in view of eqn (113) the gradient of ψ is:

$$\nabla\psi = \left(\frac{h}{2}\left(1-2\frac{N}{N^c}\right)\frac{|M|}{M},\ 1\right), \quad \text{for} \quad M \neq 0, \tag{114}$$

δ and φ must satisfy the condition

$$\delta-\varphi\frac{h}{2}\left(1-2\frac{N}{N^c}\right)\frac{|M|}{M} = 0, \tag{115}$$

and, by virtue of eqn (108), we can write:

$$\delta = -\varphi d. \tag{116}$$

So, at the instant of collapse, the sections which are hinge sites rotate round the neutral axis.

In order to calculate the internal work $\mathscr{L}_i$ for each hinge, let us consider $\mathbf{q}$ belonging to Γ and α an admissible generalised displacement. From eqns (109) and (115), recalling that $N^c = -h\sigma^c < 0$ and by assuming φ and M to have the same sign, we immediately deduce:

$$\mathscr{L}_i = \mathbf{q}\cdot\alpha = N\delta+M\varphi = \varphi\frac{Nh}{2}\left[\left(1-2\frac{N}{N^c}\right)-\left(1-\frac{N}{N^c}\right)\right]\frac{|M|}{M} = -\varphi\frac{N^2h}{2N^c}\frac{|M|}{M} = +\frac{N^2}{2\sigma^c}|\varphi|. \tag{117}$$

Figure 15 shows both the line of thrust and the two curves given in eqn (112) drawn in correspondence of the last load increment for which the convergence is reached. From these curves one immediately deduces the mechanism of collapse, and the position of the five hinges may be estimated with a close approximation. The distance of the five hinges from the mean line of the arch is determined using relation (108), the normal force in the section being already known.

Figure 16 shows the circumferential stress. The maximum level of compression is reached at the extrados, near the crown and the springing, and at the intados in the proximity of haunches; in the remaining zones the circumferential stress is nil and the material does no work.

In order to check the results of the analysis, we determine an upper bound of the collapse load by using the hinge positions and the corresponding values of the normal force obtained numerically. For symmetry reasons it sufficient to consider half of the arch, as shown in Fig. 17.

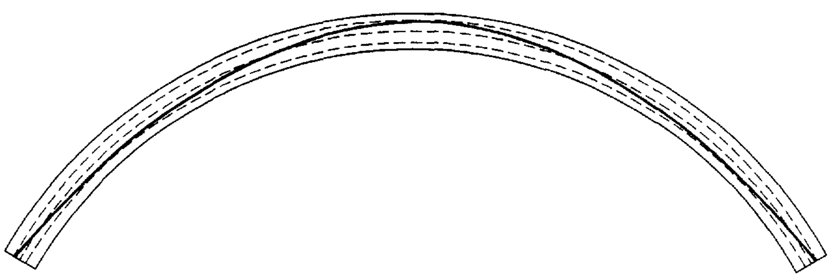

Fig. 15. The line of thrust when collapse is reached.

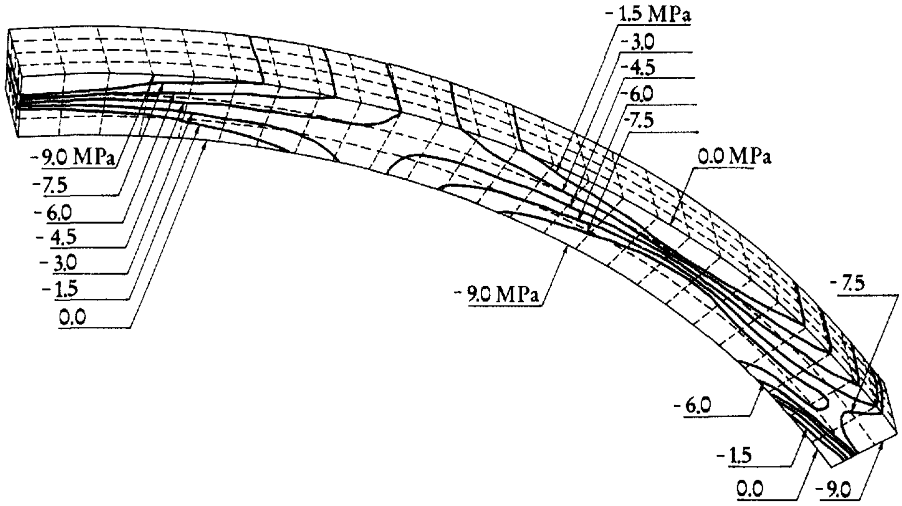

Fig. 16. Circumferential stress distribution when collapse is reached.

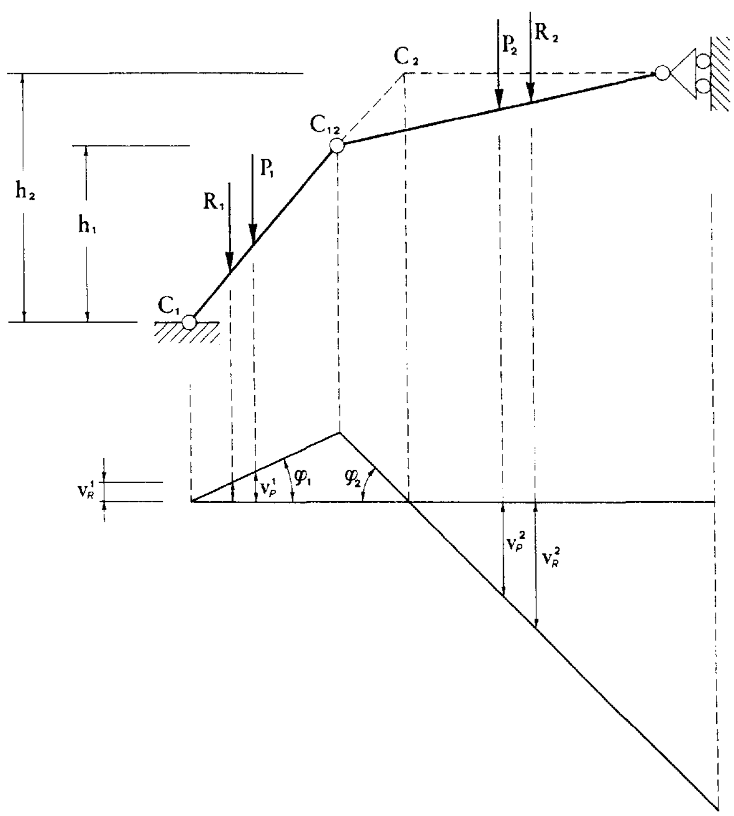

Fig. 17. Collapse mechanism.

Assigning the resultants of load and their own weight to the two sections into which the hinge C_{12} divides the half arch under consideration, the virtual work principle allows one to write

$$\mathscr{L}_{\mathrm{i}} = \mathscr{L}_{\mathrm{e}} \tag{118}$$

where, by virtue of eqn (117),

$$\mathscr{L}_{\mathrm{i}} = \frac{1}{2\sigma^{\mathrm{c}}}[N_1^2|\varphi_1| + N_2^2(|\varphi_1| + |\varphi_2|) + N_3^2|\varphi_2|], \tag{119}$$

with N_1, N_2 and N_3 the normal forces at the springing, at the hinge C_{12} and at the crown, respectively. Moreover, referring to Fig. 17, we have

$$\mathscr{L}_{\mathrm{e}} = -R_1 v_{R_1} - P_1 v_{P_1} + P_2 v_{P_2} + R_2 v_{R_2}. \tag{120}$$

Since $\varphi_2 = \varphi_1 h_2/(h_2 - h_3)$, $\mathscr{L}_{\mathrm{i}}$ and $\mathscr{L}_{\mathrm{e}}$ may be expressed as function of φ_1; the angle φ_1, in turn, may be eliminated from both sides of eqn (118). From this last relation one can obtain the value of the collapse load, which in our case is 0.42 MPa, with a 3% margin of error.

7. CONCLUSIONS

The numerical examples presented in Section 6 show that the constitutive model of BCS materials and the numerical techniques proposed in this paper, allow a realistic description of the static behaviour of masonry structures, at least in several cases having practical interest.

The generalisation made by setting a limit to the maximum compressive stress the material can sustain, even if it makes the constitutive equation more complicated, allows a more realistic evaluation of the stress field, as shown by the last numerical example. Even if the BCS material is hyperelastic, the non-linearity of the constitutive equation requires that in the numerical applications the load is assigned incrementally. On the other hand this procedure is justified by the fact that, as proven in Section 3, the stress does not depend on the particular load process.

In this model cracks arising in a body are represented by a part of the strain, namely the fracture strain. It is important to remark that in many cases the displacement field is continuous, in spite of the presence of non-nil fracture strains, as it happens in the examples presented in Section 4, where, even if the body is widely cracked, the displacement is everywhere continuously differentiable. This fact allows to use the classical finite element method which supposes the continuity of displacements.

Acknowledgements—We wish to thank P. Podio-Guidugli for his useful remarks concerning calculation of the derivative of **T** with respect to **E**.

REFERENCES

Anzellotti, G. (1985). A class of non-coercive functionals and masonry-like materials. *Ann. Inst. H. Poincaré* **2,** 261–307.

Bennati, S. and Padovani, C. (1992). Explicit solutions for equilibrium problems of masonry-like solids. Internal Report CNUCE C92-19.

Castigliano, C. A. P. (1879). *Théorie de l'equilibre des systèmes élastiques et ses applications.* Augusto Federico Negro, Turin.

Del Piero, G. (1989). Constitutive equation and compatibility of the external loads for linear elastic masonry-like materials. *Meccanica* **24,** 150–162.

Giaquinta, M. and Giusti, E. (1985). Researches on the equilibrium of masonry structures. *Arch. Rat. Mech. Anal.* **68,** 359–392.

Lucchesi, M., Padovani, C., Pagni, A. and Zani, N. (1993) Un metodo numerico per lo studio di archi in muratura. Atti del VII Convegno Italiano di Meccanica Computazionale, pp. 239–244. Trieste, 1–3 giugno.

Lucchesi, M., Padovani, C. and Pagni, A. (1994a). A numerical method for solving equilibrium problems of masonry-like solids. *Meccanica* **29,** 175–193.

Lucchesi, M., Padovani, C. and Pasquinelli, G. (1994b). On the numerical solution of equilibrium problems of elastic solids with bounded tensile strength. *Comp Meth. Appl. Mech. Engng* (in press).

Lucchesi, M., Padovani, C., Pasquinelli, G. and Zani, N. (1994c). Un metodo numerico per le volte in muratura. Atti del VIII Convegno Italiano di Meccanica Computazionale, pp. 44–49. Torino, 15–17 giugno.

Rossi, R. and Sassu, M. (1994) Masonry walls with holes: the penalty surface technique for the global response to horizontal-in-plane forces. *EUROMECH*, 2nd European Solid Mechanics Conference, Genoa, 12–16 September.